On the strain differential effect in the theory of shape memory membranes

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Abstract. A theory of thin shells made of shape memory alloys is developed. Such thin-walled elements could be used in various structures as sensors, actuators, vibration dampers, etc. The new incremental constitutive equations for a thin shape memory membrane shell are derived accounting for the strain differential effect taking the once coupled phenomenological model of thermoelastic phase transitions as a background. The solutions for spherical and cylindrical shape memory alloy shells loaded by the internal or external pressure and axial loads are obtained. It is shown that the ratio of second invariant of the deviatoric stress and the stress intensity impacts significantly on the strains accumulated by the shell during martensite phase transitions, as well as on the temperature ranges of phase transforms.

1 Introduction

Shape memory allows (referred hereinafter to as SMAs) show very specific thermomechanical behavior due to phase transition between two stable phases, the face-centred martensite (M-phase) and body-centred austenite (A-phase) accompanied with structure transitions between twinned and detwinned martensite. The phase transitions could be induced by cooling or heating as well as by raising or drop of stresses. Such specificities of the thermomechanical behavior at the micro-scale result in several effects at macro-scale, e.g. shape recovery during M → A transform (so-called “shape memory”), pseudoelastic deforming with large hysterethic loops, etc (e.g. see [1]). These effects could be used efficiently to design various devices such as sensors, actuators, and others.

Since the behavior of bulk SMA elements is well investigated using different mathematical models, the mechanics of thin-walled curved SMA elements is not yet studied in details. Some 2D models were proposed for ultra-thin single-crystal SMA films; for instance, a theory for such a film based on the nonlinear elasticity model with a supplementary interfacial energy term and Cosserat membrane kinematics was proposed in [2, 3]. Another way to derive the thin film theory may consist in the dimensional reduction of the 3D variational model of coupled thermomechanical behavior of SMA formulated in terms of Gibbs free energy and dissipating potential [4].

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Most of these theories deal with ultra thin (i.e. micron-size) SMA films on substrates that are widely used in various microelectromechanical devices, e.g. the papers [5, 6] provide some numerical results based on the finite element simulation. Let us note that not only micro- but also thin-walled macro-devices made of SMAs (e.g. reversible bellows [7]) may be useful, and appropriate modeling of such structures requires a theory of thin SMA shells. One could note some solutions based on 3D formulations for relatively thick shells (e.g. see [8, 9]). The well-developed kinematically nonlinear thin shell theory [10, 11] is based on the Gibbs concept of phase transitions that considers a sharp phase interface; unfortunately this approach is consistent only with single-crystal materials but not with most binary SMA. Indeed, two phase interfaces of start and finish of the phase transition with mixed phase constitution between them are observed experimentally (referred to as “smeared” phase transitions [12]). An appropriate macro-scale phenomenological model for SMA undergoing non-isothermal thermoelastic phase transitions was developed by A. Movchan; it is able to describe theoretically most known phenomena such as shape memory, pseudoelasticity, bounded creep, and others (e.g. see [13]) and was used in particular to study the buckling of SMA structures that occurs under abnormally low critical stresses [14-17].

Let us note that the qualitative analysis of the buckling phenomenon requires an appropriate thin shell theory that is yet missing; only one solution was obtained for the thin-walled cylinder [17]. A known difficulty that appears when such a theory is formulated using translation and rotation fields as primary variables consists in the inversion of incremental constitutive equations [13] that should be performed numerically at each point of the deformation pattern [15, 16]; the appropriate analytical formulae were proposed only recently [18]. At the same time an intrinsic formulation of the shell theory in terms of tangent and bending strains [19] could be more useful since it does not require such inversion. The intrinsic formulations based on the SMA model [13] were proposed in [20-22], the appropriate compatibility equations were derived as well for 3D bodies [20] as for thin shells [21, 22] together with the boundary conditions in terms of generalized strains [22]. Here the particular case of the thin SMA membrane is considered, and the strain differential effect is taken into account; it is shown that neglecting this effect could result in crucial overestimation of strains and deformations for thin-walled SMA structures undergoing phase transitions.

2 Basics of the once coupled theory of SMA membranes undergoing thermoelastic phase transforms

1.1 Intrinsic formulation of the thin shell theory

1.1.1 2D shell model's geometry

Let the shell model be defined on some smooth 2D manifold \( S \) immersed in the 3D euclidean space \( \mathbb{R}^3 \), and let us define on the cotangent fibration \( T^*S \) two tensor fields, the metrics \( a_{\alpha\beta} \) and the curvature \( b_{\alpha\beta} \) [21] interrelated by Petersson-Codazzi-Gauss equations; let \( S \) be a model of the shell midsurface. Thus, the shell is considered below as
a material surface $S$ furnished by some reduced mechanical properties following from the dimensional reduction of the 3D model of a body made of SMA. The deformed state of the shell modelling surface could be defined intrinsically within two covariant tensor fields on the fibration $T^*S$

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(a'_{\alpha\beta} - a_{\alpha\beta}), \quad \kappa_{\alpha\beta} = b'_{\alpha\beta} - b_{\alpha\beta}, \quad \alpha, \beta, \gamma, \delta, \ldots = 1, 2;$$

$\varepsilon_{\alpha\beta}$ is the tangent strain while $\kappa_{\alpha\beta}$ is the curvature change, $c^{\alpha\beta}$ denotes Levi-Cvita symbol. These quantities are interrelated by the compatibility equations for small strains [23]:

$$\nabla_{\beta}(c^{\beta\gamma}c^{\alpha\delta} \mu_{\gamma\delta}) = -b^\gamma_{\delta} \nabla_{\beta}(c^{\beta\gamma}c^{\alpha\delta} \varepsilon_{\gamma\delta}); \quad b_{\alpha\beta}(c^{\beta\gamma}c^{\alpha\delta} \mu_{\gamma\delta}) = \nabla_{\alpha} \nabla_{\beta}(c^{\beta\gamma}c^{\alpha\delta} \varepsilon_{\gamma\delta}),$$

$$\mu_{\gamma\delta} = -\kappa_{\gamma\delta} + b^\gamma_{\gamma} \varepsilon_{\gamma\delta}$$ is the bending strain of $S$ (for thin shells we could consider $\mu_{\gamma\delta} \approx -\kappa_{\gamma\delta}$).

### 1.1.2 3D shell model’s geometry

Let the shell be a body $V \in \mathbb{R}^3$, $\partial V = S_\beta \oplus S_\perp$ where $S_\perp \neq \emptyset$ denotes shell’s faces and $S_\beta$ corresponds to its lateral surface, and let it be referred to the curvilinear frame $O x \xi \zeta$ normally attached to the manifold $S$ [21, 22] where $\zeta \in [-h/2, h/2]$ denotes normal coordinate and $h$ is the shell thickness. Let the thickness be sufficiently small: $h/R_0 \leq 1$,

$$R_0 = \inf_{M \in S} \left[k^{-1}_a(M), \sup_{M'_1, M'_2 \in S} M'_1 M'_2\right] \left(\partial S \neq \emptyset\right), \quad R_0 = \inf_{M \in S} k^{-1}_a(M) \left(\partial S = \emptyset\right),$$

where $k_a, \alpha = 1, 2$ are principal values of $b_{\alpha\beta}$. Thus, the metrics on the fibration $T^*V$ could be expressed through the metrics $a_{\alpha\beta}$ and curvature $b_{\alpha\beta}$ as follows: $\forall M \in V$

$$g_{\alpha\beta} = a_{\alpha\beta} - 2\zeta b_{\alpha\beta} + o(h), \quad g_{\alpha 3} = 0, \quad g_{33} = 1$$ [22]. Since the 3D strain state of the shell could be determined by the Cauchy strain tensor $e_{\gamma} = \frac{1}{2}(g'_{\gamma\gamma} - g_{\gamma\gamma})$ (here prime denotes acting configuration of the 3D body $V$ or the midsurface $S$), accounting for we obtain

$$e_{\alpha\beta} = \frac{1}{2}(a'_{\alpha\beta} - a_{\alpha\beta}) - \zeta(b'_{\alpha\beta} - b_{\alpha\beta}) = e_{\alpha\beta} - \zeta \kappa_{\alpha\beta}.$$
$e_{33} \to 0$, and the transverse strain $e_{33}$ is defined hence by the plane stress condition, $\sigma_{33} \to 0$.

1.1.3 Generalized forces and equilibrium equations for the shell model

Generally the forces in the shell could be determined through the surface strain energy density

$$W = \frac{1}{2} \int_{-h/2}^{h/2} \sigma^{\alpha \beta} \sqrt{g/a} \, d\zeta \approx \frac{1}{2} \int_{-h/2}^{h/2} \sigma^{\alpha \beta} e_{\alpha \beta} d\zeta = \frac{1}{2} \left( N^{\alpha \beta} e_{\alpha \beta} + M^{\alpha \beta} \mu_{a \beta} \right) \approx \frac{1}{2} \left( N^{\alpha \beta} e_{\alpha \beta} - M^{\alpha \beta} \kappa_{a \beta} \right),$$

where the tangent force tensor $N^{\alpha \beta}$ and couple tensor $M^{\alpha \beta}$ are defined as follows [22, 23]:

$$N^{\alpha \beta} = \int_{-h/2}^{h/2} \sigma^{\alpha \beta} \sqrt{g/a} \, d\zeta \approx \int_{-h/2}^{h/2} \sigma^{\alpha \beta} d\zeta; \quad M^{\alpha \beta} = \int_{-h/2}^{h/2} \sigma^{\alpha \beta} \sqrt{g/a} \zeta d\zeta \approx \int_{-h/2}^{h/2} \sigma^{\alpha \beta} \zeta d\zeta.$$

The stress tensor components could be determined hence as $\sigma^{\alpha \beta} = h^{-1} N^{\alpha \beta} + 12 h^{-3} \zeta M^{\alpha \beta}$.

The following equations determine the elastic strains of the shell through $N^{\alpha \beta}$, $M^{\alpha \beta}$ [22, 23]:

$$\varepsilon^{E}_{\alpha \beta} \approx E^{-1} h^{-1} \left[ (1 + \nu) N_{\alpha \beta} - \nu a_{\alpha \beta} a_{\gamma \delta} N_{\gamma \delta} \right], \quad \kappa^{E}_{\alpha \beta} \approx -12 E^{-1} h^{-3} \left[ (1 + \nu) M_{\alpha \beta} - \nu a_{\alpha \beta} a_{\gamma \delta} M_{\gamma \delta} \right].$$

In general, the statics of thin shells is defined by the following equilibrium equations [23]

$$\nabla_\beta N^{\alpha \beta} - b^{\alpha \gamma} \nabla_\gamma M^{\beta \gamma} - q^\alpha = 0, \quad \nabla_\alpha \nabla_\beta N^{\alpha \beta} + b_{\alpha \beta} N^{\beta \gamma} - p = 0$$

and the boundary conditions, as well statical on the contour $\Gamma_o$ as deformational on $\Gamma_e$ [22].

Let us consider hence a membrane stress state of the shell, i.e.

$$\sigma^{00} \left( M^{00} \right) \| \sigma^{00} \left( N^{00} \right) \quad \Rightarrow \quad M^{00} \| \frac{1}{12} h^2 N^{00},$$

therefore the equilibrium equations could be simplified and reduced to the system:

$$\nabla_\beta N^{00} - p^0 \approx 0; \quad b_{0 \beta} N^{00} - p \approx 0,$$

so that the system becomes statically determinate. Such a solution could be interpreted at least as a specific solution for the general case or may be general if appropriate boundary conditions do not restrain the deformations of the shell.

1.2 Incremental constitutive equations for thin SMA shells

Let us consider a Nickel-Titanium SMA with two stable phase states, the martensite one with Young modulus $E_M$ and thermal expansion factor $\alpha_M$ and austenite one with
modulus $E_A$ and thermal expansion factor $\alpha_A$, and let these quantities be defined by mixture rules for any mixed constitution, i.e.,
\[ E^{-1}(q) = (1-q)E_A^{-1} + qE_M^{-1}, \]
\[ \alpha(q) = (1-q)\alpha_A + q\alpha_M \] [13]. Accordingly to the once coupled model described in [13] the phase constitution of SMA is defined by the martensite volume ratio $q \in [0,1]$ where $q = 0$ corresponds to the pure austenite referred hereinafter to as A-state and $q = 1$ to the pure martensite, i.e. the M-state.

1.2.1 Incremental evolution equation for a thin SMA membrane undergoing non-isothermal thermoelastic phase transforms

For a thin membrane shell we could represent this quantity by the McLaurin expansion term of zeroth order, i.e.,
\[ q(\xi^1, \xi^2, \zeta) = q^0(\xi^1, \xi^2) + o(h), \]
where
\[ q^0(\xi^1, \xi^2) = q(\xi^1, \xi^2, 0), \]
as well as the temperature
\[ T(\xi^1, \xi^2, \zeta) = T^0(\xi^1, \xi^2) + o(h), \]
\[ T^0(\xi^1, \xi^2) = T(\xi^1, \xi^2, 0) \] [22]. Since the finite relation for $q^0$ may be used only for complete phase transitions, e.g. from pure A-state up to pure M-state (so-called “complete direct martensite transform”), or vice versa (“complete inverse martensite transform”). In general, an increment of the phase constitution parameter could be defined in 3D as provided by [24] or, for a thin SMA shell, accordingly to [21, 22]:
\[ \delta q^0 = \Phi^{(0)}(q^0, T^0, N^{\alpha\beta}) \left( \delta T^0 - \Delta S^{-1} Q_{\alpha\beta}^{(0)} \delta N^{\alpha\beta} \right); \]
thus, the increment of the across-the-thickness average of the martensite volume ratio depends linearly on the given temperature (in terms of once coupled theory neglecting latent heat of the phase transition) and the tangent force tensor. Here and below the indices “+” and “−” correspond to the direct and inverse phase transitions, respectively, $\Delta S$ is the entropy difference between pure martensite and austenite phase constitutions, and for $\Phi^{(0)}$ we have
\[ \Phi^{(0)} = -\pi \frac{1-q^0}{T_S^+ - T_F^+} \tan \left[ \frac{\pi}{2} \frac{T_S^+ \left( N^{\alpha\beta} \right) - T^0}{T_S^+ - T_F^+} \right], \]
\[ \Phi^{-0} = -\pi \frac{q^0}{T_S^- - T_F^-} \cot \left[ \frac{\pi}{2} \frac{T^0 - T_S^- \left( N^{\alpha\beta} \right)}{T_F^- - T_S^-} \right]; \]
\[ T_{S,F}^\pm \] are the boundaries of the temperature ranges of the direct or inverse martensite transforms, i.e. the direct transform is possible under cooling of the SMA shell through the temperature range $[T_F^-, T_S^+]$ while the inverse one under heating through the range $[T_S^-, T_F^+]$. These temperatures depend on the deviatoric stress $\hat{\sigma}_{\alpha\beta}$ and accumulated phase strain $\hat{\varepsilon}_{\alpha\beta}$. 
Assuming the additive strain tensor decomposition, i.e.
\[ \varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta}^E + \varepsilon_{\alpha\beta}^T + \varepsilon_{\alpha\beta}^F, \]
where the elastic tangent strain is given by, \[ \varepsilon_{\alpha\beta}^T = \alpha(q^0) a_{\alpha\beta}(T^0 - T^*), \] is the temperature strain, \[ T^* \] denotes some reference temperature, and \[ \varepsilon_{\alpha\beta}^E \] is the strain accumulated after the phase transition, we could introduce the following notations in accordance with [13, 20-22]:
\[ \omega^{(0)} = \rho_D(s) \left[ 1 - q^0 f^+ (q^0) \right] \varphi_i \left( \sigma_j / \sigma_0 \right) \sigma_j + f^+ (q^0) \varepsilon_{\alpha\beta}^E \hat{\sigma}_{\alpha\beta}, \quad \sigma_i = \sqrt{\frac{1}{2} \left( \hat{\sigma}_{\alpha\beta} \hat{\sigma}_{\alpha\beta} + \sigma^2 \right)}; \]
\[ W_{\Delta} = \frac{1}{2} E^{-1}_{\Delta} h^{-2} \left[ \frac{1 + \frac{q}{2}}{2} a_{\alpha\beta} a_{\beta\alpha} + a_{\alpha\beta} a_{\beta\alpha} \right] N_{\alpha\beta} N_{\beta\alpha}, \quad E_{\Delta}^{-1} = E_{M}^{-1} - E_{A}^{-1}, \]
here \[ f^+ (q^0) = 1/(2 + q^0), \quad f^- (q^0) = 1/q^0, \quad \varphi_i (x) = \text{erf} \left( x/\sqrt{2} \right), \quad \varepsilon_0 \] is the bulk strain of the direct martensite transition, and \[ \sigma_0 \] is the threshold stress value (for more details see [13, 22]).

Let us note that the supremum of the deviatoric phase strain \[ \rho_D \] depends on the stress state; it tends to 0.1 under pure tension, is very close to 0.1 under pure shear, and drops up to 0.6 under the conditions of pure compression. Such a dependence could be approximated:
\[ \rho_D(s) = \rho_D^0 \begin{cases} 1, & s \in [0, 1] \\ 1 + \rho_D^s s, & s \in [-1, 0) \end{cases} \]
where \[ s = \frac{3}{2} \sigma_i^3 \det(\hat{\sigma}_i) = -\frac{3}{2} \sigma_i^2 \sigma h^{-2} \det(N_{\alpha\beta}^0) \] indicates the stress state type. Accounting for we have finally for the covariant tensor \[ Q_{\alpha\beta}^{(0)} \] the formula (see also [22]):
\[ Q_{\alpha\beta}^{(0)} = \omega_{\alpha\beta}^{(0)} + \omega_{\alpha\beta}^{(0)} + \omega_{\alpha\beta}^{(0)} + \Delta^{-1}/2 \hat{\sigma}_{\alpha\beta} + \varepsilon_{\alpha\beta}^E \]
\[ \omega_{\alpha\beta}^{(0)} = \frac{1}{2} \rho_D(s) h^{-1} \left[ 1 - 7 q^0 f^+ (q^0) \right] \varphi_i \left( \sigma_j / \sigma_0 \right) \sigma^{-1}_j \hat{\sigma}_{\alpha\beta} + f^+ (q^0) \varepsilon_{\alpha\beta}^E, \]
\[ \omega_{\alpha\beta}^{(0)} = \frac{1}{2} \rho_D(s) h^{-1} \left[ 1 - q^0 f^+ (q^0) \right] \varphi_i \left( \sigma_j / \sigma_0 \right) \sigma^{-1}_0 \hat{\sigma}_{\alpha\beta}, \]
\[ \omega_{\alpha\beta}^{(0)} = \frac{1}{2} \rho_D(s) h^{-1} \left[ 1 - q^0 f^+ (q^0) \right] \varphi_i \left( \sigma_j / \sigma_0 \right) S_{\alpha\beta}, \]
\[ S_{\alpha\beta} = -\frac{1}{2} \sigma_i^2 \hat{\sigma}_{\alpha\beta} + \frac{1}{2} \det(\hat{\sigma}_i) \left( \hat{\sigma}_i^\alpha \right)^{-1} c^{\alpha\beta}_{\alpha\beta} \hat{\sigma}_i^\alpha \Delta_{\alpha\beta}^\alpha + \frac{1}{3} \sigma^{-1}_{\alpha\beta}, \]
\[
\Delta_{\alpha_0\beta_0} = \frac{1}{2} \left( a_{\alpha\beta} a_{\beta\alpha} + a_{\alpha\beta} a_{\alpha\beta} \right) - \frac{1}{3} a_{\alpha\beta} a_{\alpha\beta}, \quad \rho_D'(s) = \begin{cases} 0, & s \in [0, 1]; \\ \rho_D^\lambda, & s \in [-1, 0). \end{cases}
\]

1.2.2 Incremental constitutive equation for a thin SMA membrane undergoing non-isothermal thermoelastic phase transitions

Considering for the general constitutive equations we could define both elastic and temperature tangent strains for the SMA membrane shell as follows [21, 22]:

\[
\varepsilon_{\alpha\beta}^E = h^{-1} \left( E^{-1} + q^0 E_\lambda^{-1} \right) \left[ \left( 1 + q^0 \right) N_{\alpha\beta} - \frac{2}{3} \alpha \sigma_{\alpha\beta} N \right], \quad N = \frac{1}{2} a_{\alpha\beta} N^{\alpha\beta};
\]

\[
\varepsilon_{\alpha\beta}^\lambda = \left( \alpha + q^0 \alpha \lambda \right) a_{\alpha\beta} \left( T^0 - T^0 \right), \quad \alpha = \alpha_M - \alpha_A.
\]

Since the elastic and temperature strains could be determined using finite relations \( \varepsilon_{\alpha\beta}^E \), one should introduce only a small increment of the tangent phase strain \( \delta \varepsilon_{\alpha\beta}^\pm \) [22]:

\[
\delta \varepsilon_{\alpha\beta}^\pm = \delta \varepsilon_{\alpha\beta}^E + \frac{1}{2} a_{\alpha\beta} \delta \theta^\pm, \quad \delta \varepsilon_{\alpha\beta}^\pm = \omega_{\alpha\beta}^q q^0, \quad \delta \theta^\pm = \varepsilon_0 \delta q^0
\]

where the martensite volume ratio increment is defined by the incremental equation.

Thus, the tangent strain for a thin SMA membrane shell undergoing thermoelastic pase transforms is defined by the equations \( \varepsilon_{\alpha\beta}^E \). Assuming for the compatibility of the summary strains we should compute the curvature change for a membrane using.

2 Deformation of the SMA membranes undergoing non-isothermal phase transitions

Let us consider various SMA membranes loaded by constant pressure \( P \) and cooled/heated through the temperature ranges of direct/inverse martensite transforms. We should keep in mind that SMA shells buckles under abnormally low critical stresses, and the strain differential effect given by the dependence \( \rho_D(s) \) could impact significantly on critical stress values (this effect should be studied in details).

The following material constants for the Nickel Titanium SMA are used (see Table 1):

<table>
<thead>
<tr>
<th>( E_\lambda ), GPa</th>
<th>( E_M ), GPa</th>
<th>( \nu )</th>
<th>( \rho_D^0 )</th>
<th>( \rho_D^\lambda )</th>
<th>( \varepsilon_0 )</th>
<th>( \alpha_\lambda ), K(^{-1} )</th>
<th>( \alpha_M ), K(^{-1} )</th>
<th>( \Delta S ), J/K</th>
<th>( \sigma_0 ), GPa</th>
</tr>
</thead>
<tbody>
<tr>
<td>70</td>
<td>30</td>
<td>0.31</td>
<td>0.1</td>
<td>0.4</td>
<td>10(^{-3} )</td>
<td>1.2( \times )10(^{-6} )</td>
<td>0.6( \times )10(^{-6} )</td>
<td>6.44( \times )10(^{-3} )</td>
<td>0.1</td>
</tr>
</tbody>
</table>

Table 1. Material constants for the SMA of type TN-1.
2.1 Spherical SMA shell undergoing direct/inverse martensite transforms

Let us consider a spherical SMA shell of radius \( R = 1 \text{ m} \) and thickness \( h = 0.01 \text{ m} \) loaded by the constant pressure \( p = 0.2 \text{ MPa} \). The initial temperature is equal to \( 80 + 273 \text{ K} \); the shell is cooled up to the temperature \( 15 + 273 \text{ K} \) and then heated up to the temperature \( 80 + 273 \text{ K} \). The phase state evolution is shown on the Fig. 1 whereas the dependence of the strain components \( \varepsilon_{00}^\pm (T^0) \) on the temperature is shown on the Fig. 2.

![Martensite volume ratio](image)

**Fig. 1.** Phase state evolution of the spherical shell under internal/external pressure \( p = 0.2 \text{ MPa} \).

For a spherical shell \( N_{qp} = N_{00} = pR/2 \), and the deviatoric stress components are \( \hat{\sigma}_{qp} = \hat{\sigma}_{00} = pR/(6h) \); \( \text{sgn}(\hat{\sigma}_{qp}) = \text{sgn}(\hat{\sigma}_{00}) \), due to the plane strain for a thin structure the third deviatoric stress is negative, thus, the internal pressure results in \( s = -1 \) while the external one to \( s = 1 \), so that leads to higher strain under compression. It could be seen that at this pressure value the strain differential effect is relatively small due to small stress intensity (and small value of the function \( \varphi_i(\sigma_i/\sigma_0) \)).
Let us apply the pressure $p = 2$ MPa; such a load would result evidently to shell buckling under external pressure. The results after cooling the shell from $90 + 273$ K up to $25 + 273$ K and then heating up to $90 + 273$ K are shown nevertheless on Fig.3:

It could be seen that at higher pressure the effect of strain differential effect become significant; indeed, neglecting this effect leads to $\sim 40\%$ overestimation of the tangent strain. Moreover, taking into account the strain differential effect allows one to compute the supplementary shifting of temperatures of the beginning and end of the phase transformation; for the considered problem such a shift is about 7K.
2.2 Cylindrical SMA shell undergoing direct/inverse martensite transforms

Let us consider hence a cylindrical SMA shell of radius \( R = 1 \) m and thickness \( h = 0.01 \) m loaded by the constant internal pressure \( p = 1 \) MPa. The initial temperature is equal to \( 90 + 273 \) K; the shell is cooled up to the temperature \( 25 + 273 \) K and then heated up to the temperature \( 90 + 273 \) K.

For such a loading we have \( N_\varphi = pR/2 \), \( N_\theta = pR \) (for such a problem the membrane solution is only partial one). It could be seen that for such a loading \( \hat{\sigma}_{zz} = 0 \); the dependence of the strain components \( \varepsilon_{00}^\pm(T^0) \) on the temperature is shown on the Fig. 4. Let us apply hence a compressive load along the cylinder axis, so that \( N_\varphi = (0.9 \sigma)p/2 \). The longitudinal deviatoric stress component become negative, \( \hat{\sigma}_{zz} < 0 \). The dependence of the strain components \( \varepsilon_{00}^\pm(T^0) \) on the temperature is shown on the Fig.5, and the obtained results are compared on the Fig. 6.

![Phase strain tensor components](image)

**Fig. 4.** Phase strains in the cylindrical shell under internal pressure \( p = 1 \) MPa.
Fig. 5. Phase strains in the cylindrical shell under internal pressure $p = 1$ MPa: drop in longitudinal force due to the supplementary compression $0.1pR/2$.

Fig. 6. Circumferential phase strains $\varepsilon_{\phi \theta}^+\varepsilon_{\phi \theta}^-$ in the shell with or without longitudinal compression.

3 Conclusion

- The new evolution and constitutive equations for thin shape memory alloy shells are constructed accounting for the strain differential effect considered by introducing a piecewise-linear approximation for the dependence of the supremum of the deviatoric phase strain on the ratio of the second invariant of the deviatoric stress in a shell and the stress intensity.
- The obtained incremental constitutive equations are solved for shell tangent strains, thus, the so-called “intrinsic” formulation of the shell model becomes less difficult than the traditionally used formulation based on the translation components as primary variables; indeed, the intrinsic formulation does not require any inversion of the constitutive equations.
The significant dependence of the phase strains accumulated after austenite-to-martensite phase transitions on the second invariant of the deviatoric stress is observed under higher loads; it may reach 40%.

The internally pressurized convex shells with both positive tangent force tensor components show less phase compliance than compressed shells; such an effect could be significant for midsurfaces with negative Gaussian curvatures and should impact on stability of equilibrium of shells undergoing phase transforms.

The strain differential effect results in the supplementary shifting in the temperatures of beginning and end of phase transitions; neglecting this effect could result in overestimation of the temperature of start of the direct phase transform, etc.

Due to the stress differential effect the small load variation for stress states with one deviatoric stress tending to zero may result in significant variation in strain amplitudes, so that may also impact on the equilibrium stability.

The stress differential effect could be neglected for slightly loaded shells with phase strain amplitudes about 10% of the maximum possible value and should be accounted for shells with positive stresses close to the yield stress of SMA.

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References


