Neumann Problem for Second-Order Differential Equation with Fractional Derivative in the Analysis and Modeling of Structures Made of Viscoelastic Elements

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Abstract. This article addresses a second-order differential equation containing a Gerasimov-Kaputo fractional differentiation operator of order less than two. The Neumann problem is formulated for this equation. A system of eigenfunctions and eigenvalues for the considered homogeneous boundary problem of the second kind is found. A conjugate boundary problem for the Gerasimov-Kaputo fractional derivative is introduced. A biorthogonal system is obtained that is orthogonal to the found system of eigenfunctions. Visualizations of the eigenfunction system, biorthogonal system, and an example of eigenvalue distribution on the real axis are provided.

1 Introduction

Second-order differential equations containing fractional derivatives are currently frequently employed in constructing mathematical models for viscoelastic materials [1-8]. A comprehensive review of mathematical models of this kind, including their historical development, can be found in works [2, 9 – 10]. Viscoelastic materials are actively utilized in various construction fields, including the analysis of reinforced concrete structures [11]. For instance, the dynamic stability of a compressed reinforced concrete element, functioning as a viscoelastic rod, is calculated [12].

Consider a second-order differential equation used to model the deformational and strength characteristics of various viscoelastic materials [13 – 19]. This second-order differential equation involves a differential operator of fractional order. Thus, the equation employed as the model of viscoelasticity (Linear Fractionally Damped Oscillator) takes the form:

\[ z''(x) + c D^\alpha_{0x} z(x) + b z(x) = f(x); \quad x \in [0; X] \]  

Here, \( z(x) \) - represents the displacement of a fixed point (a structural viscoelastic material) with respect to the abscissa \( x \);
\( f(x) \) – denotes an external influence;

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$D_{ox}^\alpha z(x)$ stands for the Gerasimov-Kaputo fractional differentiation operator of order $\alpha$, where $0 < \alpha < 2$. The definition of the fractional differentiation operator is provided as follows [20 – 22] (left-sided Gerasimov-Kaputo fractional derivative):

$$D_{ox}^\alpha z(x) = \left\{ \frac{1}{\Gamma(2-\alpha)} \int_0^x (x - \tau)^{1-\alpha} z''(\tau) d\tau, 1 << 2 \quad \frac{1}{\Gamma(1-\alpha)} \int_0^x (x - \tau)^{-\alpha} z'(\tau) d\tau, 0 << 1 \right\}$$

Here, and subsequently, $\Gamma$ represents the gamma function.

The constants $c, b, \alpha$ are presumed to be constants. According to reference [13] these constants carry the following physical meanings, which are adhered to in this study:

$c$ – material viscosity modulus,
$b$ – material stiffness modulus,
$\alpha$ – viscoelastic material parameter.

Similar to ordinary differential equations, the general solution of a nonhomogeneous equation is a sum of the general solution of the corresponding homogeneous equation and a particular solution of the nonhomogeneous equation.

Consider the corresponding homogeneous equation to equation (1):

$$z''(x) + c D_{ox}^\alpha z(x) + b z(x) = 0; \quad x \in [0; X] \quad (2)$$

When solving the Sturm-Liouville problem for equation (2), various types of boundary conditions can be set. The solution to this problem with homogeneous boundary conditions of the first kind (Dirichlet problem) is presented in [23], while mixed homogeneous boundary conditions are addressed in [24]. This work deals with solving the Neumann problem, that is, solving equation (2) under homogeneous boundary conditions of the second kind:

$$z'(0) = 0; \quad z'(X) = 0 \quad (3)$$

In other words, the inclination angle to the positive direction of the abscissa axis is zero at both the initial and final points of the viscoelastic body.

2 Methods

The first stage in solving the Neumann problem, as in any Sturm-Liouville problem, involves finding the general solution to the homogeneous equation. The general solution of equation (2) was obtained using a method outlined, for example, in [25], through solving a second-kind Volterra equation using a sequence of recurrent kernels. This solution is presented in [24] in the form of a convergent power series.

Hence, the general solution of equation (2) for the case of the Gerasimov-Kaputo fractional derivative with $0 < \alpha < 2$, in the form of a power series, takes the following form:

$$z(x) = B \left\{ x - \frac{bx^3}{6} + \sum_{n=1}^{\infty} (-1)^{n+1} \sum_{k=0}^{n} \frac{(n_k)c^k b^{n+1-k} x^{2n+3-k}}{\Gamma(2n+4-k)} \right\} +$$

$$+ c \left\{ 1 - \frac{bx^3}{6} + \sum_{k=1}^{n} (-1)^{n+1} \sum_{k=0}^{n} \frac{(n_k)c^k b^{n+1-k} x^{2n+3-k}}{\Gamma(2n+4-k)} \right\}$$

Here, $(n_k) = \frac{n!}{k!(n-k)!}$ the number of combinations from $n$ choose $k$.

The second stage of solving the Neumann problem involves applying the boundary conditions (3), resulting in the determination of the eigenvalues and eigenfunctions of the problem (2) – (3). In cases where the eigenfunction system is complete but not orthogonal over the interval $x \in [0; X]$, the construction of a biorthogonal system is required. In this work, the approach from [26] and the methods presented in [24] are employed to construct the biorthogonal system.
3 Results

For implementing the second stage of solving the Neumann problem, let's differentiate the function $z(x)$:

$$
\dot{z}(x) = B \left\{ 1 - \frac{bx^2}{2} + \sum_{n=1}^{\infty} (-1)^{n+1} \sum_{k=0}^{n} \frac{(nk)c^{k}b^{n+1-k}x^{2n+2-k}}{\Gamma(2n+3-k)} \right\} + \\
+ C \left\{ -bx + \sum_{n=1}^{\infty} (-1)^{n+1} \sum_{k=0}^{n} \frac{(nk)c^{k}b^{n+1-k}x^{2n+2-k}}{\Gamma(2n+2-k)} \right\}
$$

Using the first boundary condition:

$$
z(0) = B = 0.
$$

Then, applying the second boundary condition:

$$
z(X) = C \left\{ -bX + \sum_{n=1}^{\infty} (-1)^{n+1} \sum_{k=0}^{n} \frac{(nk)c^{k}b^{n+1-k}X^{2n+2-k}}{\Gamma(2n+2-k)} \right\} = 0
$$

From the last expression, it follows that the eigenvalues of the problem (2) – (3) are the zeros of the function $G(\lambda)$:

$$
G(\lambda) = X + \sum_{n=1}^{\infty} (-1)^{n} \sum_{k=0}^{n} \frac{(nk)c^{k}b^{n+1-k}X^{2n+2-k}}{\Gamma(2n+2-k)}
$$

In Figure 1, the graph of function $G(\lambda)$ is shown for $X = 0.5, c = 1.3, c = 0.16$.

![Graph of function](image)

**Fig. 1.** Graph of function $G(\lambda)$ for $X = 0.5, c = 1.3, c = 0.16$

Table 1 provides the first eight eigenvalues of the problem (2) – (3) for different fractional derivative orders, calculated for $X = 0.5, c = 0.16$, using formula (4) with MATLAB software (partial sum of the series was taken instead of the full sum). Accuracy - one decimal place.

**Table 1.** First 8 eigenvalues of the problem (2) – (3) for different fractional derivative orders at $X = 0.5, c = 0.16$

<table>
<thead>
<tr>
<th>λ₁</th>
<th>λ₂</th>
<th>λ₃</th>
<th>λ₄</th>
<th>λ₅</th>
<th>λ₆</th>
<th>λ₇</th>
<th>λ₈</th>
</tr>
</thead>
<tbody>
<tr>
<td>39.3</td>
<td>39.3</td>
<td>39.3</td>
<td>40.1</td>
<td>42.3</td>
<td>42.3</td>
<td>42.3</td>
<td>42.3</td>
</tr>
</tbody>
</table>
The system of eigenfunctions for the problem (2) – (3) takes the form:

\[ Z_m(x) = 1 - \frac{\lambda_m^2 x^2}{2} + \sum_{n=1}^{\infty} (-1)^{n+1} \sum_{k=0}^{n} \frac{(n+k)!}{\Gamma(2n+3-k)} \frac{\lambda_m^{n-k} x^{2n+2-k}}{2n+2-k} \]  

(5)

\(-\) are the eigenvalues; \(m = 1, 2, \ldots\)

In Figure 2, the graphs of the first four eigenfunctions for the problem (2) – (3) are presented for the fractional derivative order \(X = 0.5\), \(c = 0.16\). Similarly, Figure 3 illustrates the graphs of the eigenfunctions \(Z_5(x) - Z_8(x)\) for the problem (2) – (3) for the fractional derivative order \(X = 0.5\), \(c = 0.16\). The values of functions \(Z_1(x) - Z_8(x)\) were computed using MATLAB software through numerical modeling (partial sum of the series was used instead of the full sum).

<table>
<thead>
<tr>
<th>(\lambda)</th>
<th>157.7</th>
<th>157.5</th>
<th>157.7</th>
<th>159.7</th>
<th>167.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\lambda_2)</td>
<td>355.1</td>
<td>354.9</td>
<td>354.9</td>
<td>358.5</td>
<td>375.7</td>
</tr>
<tr>
<td>(\lambda_3)</td>
<td>631.5</td>
<td>631.1</td>
<td>631.3</td>
<td>636.3</td>
<td>665.1</td>
</tr>
<tr>
<td>(\lambda_4)</td>
<td>986.7</td>
<td>986.3</td>
<td>986.5</td>
<td>993.1</td>
<td>1036.3</td>
</tr>
<tr>
<td>(\lambda_5)</td>
<td>1420.9</td>
<td>1420.5</td>
<td>1420.5</td>
<td>1421.1</td>
<td>1488.7</td>
</tr>
<tr>
<td>(\lambda_6)</td>
<td>1934.3</td>
<td>1933.7</td>
<td>1933.7</td>
<td>1944.1</td>
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<tr>
<td>(\lambda_7)</td>
<td>2526.3</td>
<td>2525.9</td>
<td>2525.7</td>
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<td>354.9</td>
<td>354.9</td>
<td>358.5</td>
<td>375.7</td>
</tr>
</tbody>
</table>

**Fig. 2.** Graphs of eigenfunctions \(Z_1(x) - Z_4(x)\) for the problem (2) – (3) with fractional derivative order \(X = 0.5\), \(c = 0.16\)
Fig. 3. Graphs of eigenfunctions $Z_5(x) - Z_8(x)$ for the problem (2) – (3) with fractional derivative order $\alpha = 1.3$ with $X = 0.5$, $c = 0.16$

From Figures 2 and 3, the following primary oscillatory properties of the system of eigenvalues and eigenfunctions can be observed:

1) All eigenvalues are real and positive;
2) The fundamental mode does not have nodes;
3) The $m$-th overtone has exactly $i$ nodes;
4) The nodes of two consecutive overtones alternate.

In Figure 4, the graphs of the third eigenfunction for the problem (2) – (3) are presented for different fractional derivative orders: $\alpha = 0.5$, $\alpha = 1.3$ and $\alpha = 1.7$ with $X = 0.5$, $c = 0.16$. Similarly, Figure 5 illustrates the graphs of the eighth eigenfunction for the same fractional derivative orders and parameters. The values of functions $Z_3(x)$ and $Z_8(x)$ were computed using MATLAB software through numerical modeling (partial sum of the series was used instead of the full sum). Figures 4 and 5 clearly demonstrate that as the fractional derivative order increases, the extreme values of the eigenfunctions decrease.
Fig. 4. Graphs of eigenfunction $Z_3(x)$ for the problem (2) – (3) with different fractional derivative orders $\alpha = 0.5$, $\alpha = 1.3$, and $\alpha = 1.7$, $X = 0.5$, $c = 0.16$

Fig. 5. Graphs of eigenfunction $Z_8(x)$ for the problem (2) – (3) with different fractional derivative orders $\alpha = 0.5$, $\alpha = 1.3$, and $\alpha = 1.7$, $X = 0.5$, $c = 0.16$

The system of eigenfunctions $\{Z_m(x)\}_{m=1}^{\infty}$ (5) is complete in $L^2$, but not orthogonal [23]. To obtain the system $\{Z_m(x)\}_{m=1}^{\infty}$, which is biorthogonal to the system $\{Z_m(x)\}_{m=1}^{\infty}$, we turn to the adjoint problem to (2) – (3).

The adjoint boundary problem to (2) – (3) on the interval $\in [0; X]$, when using the definition of the Gerasimov-Caputo fractional differentiation operator, takes the form:

$$\ddot{z}(x) + cD_{x\xi}^{\alpha} \dot{z}(x) + b\dot{z}(x) = 0, \quad \alpha \in (0; 2)$$

(6)
It can be verified that the eigenvalues of the original and adjoint problems coincide. In turn, the eigenfunctions of the adjoint problem have the form:

$$Z_m(x) = 1 - \frac{(X-x)^2}{2} + \sum_{n=1}^{\infty} (-1)^{n+1} \sum_{k=0}^{n} \frac{(n+k)^{n-1-k}(X-x)^{2n+2-k}}{\Gamma(2n+3-k)}$$

In Figure 6, the graphs of the first six eigenfunctions for the problem (6) – (7) are presented for a fractional derivative order $= 1.3$ with $X = 0.5$, $c = 0.16$.

![Graphs of eigenfunctions](image)

**Fig. 6.** Graphs of eigenfunctions $Z_1(x) - Z_6(x)$ for the problem (6) – (7) with fractional derivative order $= 1.3$ with $X = 0.5$, $c = 0.16$

4 Discussion

Thus, any solution of the problem (2) – (3) can be represented as:

$$z(x) = \sum_{m=1}^{\infty} A_m Z_m(x)$$

where the coefficients $A_m$ are determined by the following formula:

$$A_m = \frac{(z Z_m)}{(Z_m Z_m)}.$$ (10)

Here, the scalar product in $L_2$ is used:

$$\langle w, g \rangle = \int_0^x w(x) \cdot g(x) dx.$$ (11)

For instance, when obtaining experimental data corresponding to the conditions of the solved problem, the expansion (9)-(10) can be used to find the solution of the equation (1)

$$z(x) = \sum_{m=1}^{\infty} \frac{\langle z Z_m \rangle}{\langle Z_m Z_m \rangle} Z_m(x) + z_f(x).$$
where \( \sum_{m=1}^{\infty} \frac{\langle z_{m} \rangle}{\langle z_{m}^{2} \rangle} z_{m}^{2} \) represents the general solution of the corresponding homogeneous equation, and \( z_{f}(x) \) – is any particular solution of the nonhomogeneous equation.

\section{Conclusions}

In this study, the homogeneous Neumann problem for a second-order differential equation containing the Gerasimov-Caputo fractional differentiation operator has been analyzed.

A system of eigenfunctions and eigenvalues has been obtained for the considered second-order homogeneous boundary problem.

The adjoint boundary problem for the Gerasimov-Caputo fractional derivative has been presented.

A biorthogonal system has been derived, orthogonal to the previously found system of eigenfunctions.

Visualizations of the eigenfunction system and the biorthogonal system, as well as an example of the distribution of eigenvalues along the real axis, have been provided.

The obtained results can be applied in modeling the behavior of viscoelastic materials, particularly those used in the construction industry.

\section{References}

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24. L.V. Kiryanova, T.A. Matseevich, Axioms, 12, 779 (2023) DOI:10.3390/axioms12080779