Stress-strain state of a plate located on an inhomogeneous base

A.N. Leontiev¹, K.V. Balandina, O.D. Homenko

Moscow State University of Civil Engineering, 26, Yaroslavskoye shosse, Moscow, 129337, Russia

Abstract. At present, the problems of calculating structures on an elastic foundation are very relevant. In this regard, the influence of the inhomogeneity of the base on the stress-strain state of thin plates is investigated. Plates located on the Winkler elastic foundation and having hinged supports along the contour are considered. It is assumed that the value of the foundation bed coefficient has a symmetrical character of change relative to the middle of the plate, and within its limits is described by a quadratic law. The Ritz-Timoshenko variational method was used to study the stress-strain state of the plate. The calculation was performed in the Excel environment, while trigonometric functions were taken as approximating ones. Plates with different relative lengths are presented: short, medium length and long. It is shown that the effect of base inhomogeneity on the stress-strain state is most noticeable for long plates.

Keywords. thin plates, elastic inhomogeneous foundation, Winkler model, Ritz-Timoshenko method, bed coefficient.

1 Introduction

The development of the theory of calculation of beams and slabs located on an elastic foundation can contribute to a more efficient and economical method of building various types of structures. In most works, the base is assumed to be homogeneous, however, in practice, one can often observe a change in the bed coefficient along the length of the structure, which undoubtedly affects its stress-strain state. The most common model describing the properties of an elastic foundation is the Winkler model [1, 2], according to which it is assumed that there is a directly proportional relationship between the load \( q_0(x) \) acting on the foundation and its offset \( v(x) \) of the form: \( q_0(x) = k_0 \cdot v(x) \), where \( k_0 \) is the proportionality factor, called the bed factor. It is known that under the action of a uniformly distributed load, slabs lying freely on a homogeneous elastic foundation, in accordance with the Winkler model, behave like a stamp, i.e. they lack internal effort. Obviously, even a slight inhomogeneity of the base will cause the appearance of bending moments and transverse forces. The purpose of the work is to analyze the effect of base inhomogeneity on the stress-strain state of plates of various lengths located on an inhomogeneous base.
2 Methods

A relatively thin rectangular slab with hinged support along the contour, located on a non-uniform elastic Winkler foundation, is considered. In the future, in accordance with the technical theory of Kirchhoff, we will call it a plate.

It is assumed that the value of the coefficient of the bed of the elastic foundation has a symmetrical nature of change relative to the middle of the plate, and within its limits is described by a quadratic law.

We place the origin of coordinates under the center of the plate and consider two possible options for changing the bed coefficient.

For the first case, we assume that the bed coefficient of the elastic foundation is symmetrical about one of the axes, for example, the \( y \) axis and changes along the \( x \) coordinate according to a quadratic law (Fig. 1. a):

\[
k(x, y) = \frac{\alpha}{10} \frac{12k_0}{a^2} x^2 + \frac{10 - \alpha}{10} k_0.
\]

(1)

where: \( \alpha \) – parameter characterizing the degree of heterogeneity of the base,
\( k_0 \) – average value of bed coefficient,
\( a \) – plate side length along the \( x \)-axis.

It can be shown that in this case the double integral of the function (1) does not depend on the degree of inhomogeneity of the base and coincides with a similar parameter for a uniform base, indeed:

\[
\int_{-a/2}^{a/2} \int_{-b/2}^{b/2} k(x, y) \, dx \, dy = \int_{-a/2}^{a/2} \left( \frac{\alpha}{10} \frac{12k_0}{a^2} x^2 + \frac{10 - \alpha}{10} k_0 \right) \, dx \int_{-b/2}^{b/2} \, dy = \left( \frac{\alpha}{10} k_0 a + \frac{10 - \alpha}{10} k_0 a \right) b = k_0 ab,
\]

where \( b \) – plate side length along the \( y \)-axis.

Fig. 1. The nature of the change in the bed coefficient under the plate.
\( a \) – the first case, \( b \) – the second case.
In our case, we will assume that the parameter $\alpha$, changes within $0 \leq \alpha \leq 10$. So, at $\alpha = 0$ (Fig. 2) the base is homogeneous, and at $\alpha = 10$ (Fig. 3 $a$, $b$) the heterogeneity takes on the most noticeable character. By varying the value of the parameter $\alpha$, one can obtain a more or less pronounced degree of base inhomogeneity.

In the second case, we assume that the bed coefficient of the elastic foundation is symmetrical with respect to the origin of coordinates and is described by the equation of an elliptic paraboloid (Fig. 1, $b$):

$$k(x, y) = \frac{\alpha}{10} \frac{6k_0}{a^2} x^2 + \frac{\alpha}{10} \frac{6k_0}{b^2} y^2 + \frac{10 - \alpha}{10} k_0$$  \hspace{0.5cm} (2)

In this case, the double integral of function (2) also does not depend on the degree of inhomogeneity of the base and coincides with a similar parameter for a uniform base:

$$\int_{-a/2}^{a/2} \int_{-b/2}^{b/2} k(x, y) dx dy = \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \left( \frac{\alpha}{10} \frac{6k_0}{a^2} x^2 + \frac{\alpha}{10} \frac{6k_0}{b^2} y^2 + \frac{10 - \alpha}{10} k_0 \right) dx dy =$$

$$= \frac{\alpha}{10} \frac{6k_0}{a^2} a^3 + \frac{\alpha}{10} \frac{6k_0}{b^2} b^3 + \frac{10 - \alpha}{10} k_0 ab = \left( \frac{\alpha}{20} + \frac{\alpha}{20} + \frac{10 - \alpha}{10} \right) k_0 ab = k_0 ab.$$

Here, the parameter $\alpha$, which characterizes the degree of heterogeneity of the base, again changes within $0 \leq \alpha \leq 10$ (Fig. 3, $b$).
Fig. 3. The nature of the change in the bed coefficient under the plate at $\alpha = 10$.

$a$ – the first case, $b$ – the second case.

To solve the problem, we will use the Ritz-Timoshenko variational method [3]. In the general case of transverse bending of a plate lying on an elastic foundation, the total energy $T$ is determined by the formula:

$$T = \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \left\{ \frac{1}{2} D \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + 2\nu \left( \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \right) + \frac{\partial^2 w}{\partial y^2} \right\} + k(x, y) \frac{w^2}{2} dx dy - \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} q w dx dy$$

(3)

where $D = \frac{E h^3}{12(1-\nu^2)}$ – plate cylindrical stiffness,

$w(x, y)$ – function describing vertical displacements of plate points,

$h$ – plate thickness,

$E, \nu$ – respectively, the modulus of elasticity and Poisson's ratio of the plate material.

We will look for a deflection in the form of a series:

$$w(x, y) = c_1 \varphi_1(x, y) + c_2 \varphi_2(x, y) + c_3 \varphi_3(x, y) + \ldots,$$

(4)

We substitute series (4) into the energy expression (3):

$$T = \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} \left\{ \frac{1}{2} D \left[ c_1^2 \frac{\partial^2 \varphi_1}{\partial x^2}^2 + c_2^2 \frac{\partial^2 \varphi_2}{\partial x^2}^2 + \ldots \right] + 2\nu \left[ c_1^2 \frac{\partial^2 \varphi_1}{\partial x^2} \frac{\partial^2 \varphi_2}{\partial y^2} + c_2^2 \frac{\partial^2 \varphi_2}{\partial x^2} \frac{\partial^2 \varphi_2}{\partial y^2} + \ldots \right] \right\} + k(x, y) \frac{1}{2} \left[ c_1 \varphi_1(x, y) + c_2 \varphi_2(x, y) + \ldots \right]^2 dx dy -$$

$$- \int_{-a/2}^{a/2} \int_{-b/2}^{b/2} q \left[ c_1 \varphi_1(x, y) + c_2 \varphi_2(x, y) + \ldots \right] dx dy$$
and write the condition for the minimum potential energy (Lagrange condition):

\[ \frac{\partial T}{\partial c_i} = 0. \]

\[
\frac{\partial T}{\partial c_i} = \frac{a}{\alpha} \int_0^b \int_0^b D \left[ \left( c_1 \frac{\partial^2 \phi}{\partial x^2} + c_2 \frac{\partial^2 \phi}{\partial x^2} + \ldots \right) \frac{\partial^2 \phi_i}{\partial x^2} + 2\nu \frac{\partial^2 \phi_i}{\partial y^2} \left( c_1 \frac{\partial^2 \phi}{\partial y^2} + c_2 \frac{\partial^2 \phi}{\partial y^2} + \ldots \right) \right] + \\
+ 2\nu \left( c_1 \frac{\partial^2 \phi_i}{\partial x^2} + c_2 \frac{\partial^2 \phi_i}{\partial x^2} + \ldots \right) \frac{\partial^2 \phi_i}{\partial y^2} + \left( c_1 \frac{\partial^2 \phi_i}{\partial y^2} + c_2 \frac{\partial^2 \phi_i}{\partial y^2} + \ldots \right) \frac{\partial^2 \phi_i}{\partial y^2} \right] + \\
+ k(x, y) \left[ c_1 \phi_1 + c_2 \phi_2 + \ldots \right] \Phi_i \right] \, dx \, dy - \int_0^b \int_0^b q \phi_i (x, y) \, dx \, dy = 0
\]

\[
\sum_{j=1}^k a_{ij} c_j = b_i
\]

We obtain a system of equations: \( j = 1 \), which in matrix-vector form can be written as:

\[
A \vec{c} = \vec{b}
\]

The coefficients of the system (5) are determined by the expression:

\[
a_{ij} = \frac{a}{\alpha} \int_0^b \int_0^b D \left[ \left( \frac{\partial^2 \phi_j}{\partial x^2} + \frac{\partial^2 \phi_i}{\partial x^2} \right) + 2\nu \left( \frac{\partial^2 \phi_j}{\partial y^2} + \frac{\partial^2 \phi_i}{\partial y^2} \right) \right] + k(x, y) \phi_j \phi_i \right] \, dx \, dy
\]

The elements of the right part of the system (5) are determined by integration:

\[
b_i = \int_0^b \int_0^b q \phi_i (x, y) \, dx \, dy
\]

We substitute expression (3) for \( k(x) \) into formula (6):

\[
a_{ij} = \frac{a}{\alpha} \int_0^b \int_0^b D \left[ \left( \frac{\partial^2 \phi_j}{\partial x^2} + \frac{\partial^2 \phi_i}{\partial x^2} \right) + 2\nu \left( \frac{\partial^2 \phi_j}{\partial y^2} + \frac{\partial^2 \phi_i}{\partial y^2} \right) \right] + \\
+ \left( \frac{\alpha}{10} \frac{6k_0}{a^2} x^2 + \frac{\alpha}{10} \frac{6k_0}{b^2} y^2 + \frac{10 - \alpha}{10} k_0 \right) \phi_j \phi_i \right] \, dx \, dy.
\]
We take out the cylindrical stiffness $D$ from under the sign of the integral, by analogy with the beam located on the elastic base of Winkler, we introduce the notation:

$$
\lambda^4 = \frac{k_0}{4D},
$$

and for the coefficients (6) we obtain the expression:

$$
a_{ij} = D \int \int \left\{ \frac{\partial^2 \varphi_j}{\partial x^2} \frac{\partial^2 \varphi_i}{\partial x^2} + 2\nu \left( \frac{\partial^2 \varphi_j}{\partial y^2} \frac{\partial^2 \varphi_i}{\partial x^2} + \frac{\partial^2 \varphi_j}{\partial x^2} \frac{\partial^2 \varphi_i}{\partial y^2} \right) + \frac{\partial^2 \varphi_j}{\partial y^2} \frac{\partial^2 \varphi_i}{\partial y^2} \right\} +
\frac{4\lambda^4}{10} \left( \frac{6\alpha}{a^2} x^2 + \frac{6\alpha}{b^2} y^2 + 10 - \alpha \right) \varphi_j \varphi_i \, dx \, dy
$$

In accordance with the boundary conditions of the hinge support, we take the following functions as approximating:

$$
\varphi_{ij}(x, y) = \cos\frac{i\pi x}{a} \cos\frac{j\pi y}{b},
$$

where $i, j = 1, 2, \ldots$

Then the desired deflection will be written as:

$$
\psi(x, y) = c_1 \cos\frac{\pi x}{a} \cos\frac{\pi y}{b} + c_{12} \cos\frac{\pi x}{a} \cos\frac{2\pi y}{b} + c_{21} \cos\frac{2\pi x}{a} \cos\frac{\pi y}{b} + c_{22} \cos\frac{2\pi x}{a} \cos\frac{2\pi y}{b} + \ldots,
$$

and for bending moments we get the expressions:

$$
M_x = -D \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) =
\left\{ \frac{\pi^2}{a^2} + \frac{\pi^2}{b^2} \right\} \cos\frac{\pi x}{a} \cos\frac{\pi y}{b} + \left( \frac{\pi^2}{a^2} + \nu \frac{4\pi^2}{b^2} \right) \cos\frac{\pi x}{a} \cos\frac{2\pi y}{b} +
\left( \frac{4\pi^2}{a^2} + \nu \frac{4\pi^2}{b^2} \right) \cos\frac{2\pi x}{a} \cos\frac{\pi y}{b} + \ldots
$$

Given the nature of the inhomogeneity of the base, we take in solution (9) one term of a series of the form:

$$
\psi(x, y) = c_1 \cos\frac{\pi x}{a} \cos\frac{\pi y}{b}.
$$

We will substitute functions (12) in the expression (8):

$$
a_{11} = D \int \int \left\{ \left( \frac{\pi}{a} \right)^4 + 4\nu \left( \frac{\pi}{b} \right)^4 \left( \frac{\pi}{a} \right)^2 + \left( \frac{\pi}{b} \right)^4 \right\} \cos\frac{\pi x}{a} \cos\frac{\pi y}{b} +
\frac{4\lambda^4}{10} \left( \frac{6\alpha}{a^2} x^2 + \frac{6\alpha}{b^2} y^2 + 10 - \alpha \right) \cos\frac{\pi x}{a} \cos\frac{\pi y}{b} \, dx \, dy
$$
Calculating the integrals included in expression (13), we obtain:

\[ a_{11} = \frac{D}{a^4} \frac{ab}{4} \left\{ \pi^4 \left[ 1 + 4\nu \left( \frac{a}{b} \right)^2 + \left( \frac{a}{b} \right)^4 \right] + \frac{2(\lambda a)^4}{5} \left( 10 - \frac{6\alpha}{\pi^2} \right) \right\} \]  

(14)

\[ b_i = \int_0^a \int_0^b q \cos \frac{\pi x}{a} \cos \frac{\pi y}{b} \, dx \, dy = \frac{4ab}{\pi^2} \]  

(15)

The coefficient \( c_1 \) included in expression (12) is determined by the formula:

\[ c_1 = \frac{1}{\pi^4} \left[ 1 + 4\nu \left( \frac{a}{b} \right)^2 + \left( \frac{a}{b} \right)^4 \right] + \frac{2(\lambda a)^4}{5} \left( 10 - \frac{6\alpha}{\pi^2} \right) \frac{16qa^4}{D\pi^2} \]  

(16)
3 Results

The calculation according to the obtained formulas was carried out in the Excel environment.

First of all, it should be noted that, depending on the physical and geometric characteristics of the base and the plate located on it, there are short, medium length and long plates. In this case, the assignment of the plate to one or another category depends on the value of the product \( \lambda a \).

Plates for which the product \( \lambda a < 1.3 \) are classified as short.

At the same time, plates are conventionally considered infinitely long for which \( \lambda a > 1.5 \cdot \pi = 4.71 \), and therefore the plates for which the values \( \lambda a \) are smaller, but close to this value, belong to the category of long.

Two categories of square plates are considered for comparison, while a plate with an average length is taken as a plate with \( \lambda a = 2.68 \), and as a long – plate with \( \lambda a = 4.52 \), what is determined by the specific geometric and physical parameters of these plate.

In all cases, the plates were loaded with a uniformly distributed load and the effect of the parameter \( \alpha \) in the range of variation \( 0 \leq \alpha \leq 10 \) on the values of deflection and bending moment in the center of the square plate was investigated.

Table 1 shows the results obtained for two types of plates.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( u(a/2) ) (cm)</th>
<th>( M(a/2) ) (kHm)</th>
<th>( k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>1,30</td>
<td>25,61</td>
<td>1,00</td>
</tr>
<tr>
<td>2.0</td>
<td>1,36</td>
<td>26,92</td>
<td>1,05</td>
</tr>
<tr>
<td>4.0</td>
<td>1,44</td>
<td>28,38</td>
<td>1,11</td>
</tr>
<tr>
<td>6.0</td>
<td>1,52</td>
<td>29,99</td>
<td>1,17</td>
</tr>
<tr>
<td>8.0</td>
<td>1,61</td>
<td>31,81</td>
<td>1,24</td>
</tr>
<tr>
<td>10.0</td>
<td>1,71</td>
<td>33,85</td>
<td>1,32</td>
</tr>
</tbody>
</table>

Results for medium length plate (\( \lambda a = 2.68 \))

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( u(a/2) ) (cm)</th>
<th>( M(a/2) ) (kHm)</th>
<th>( k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>2,73</td>
<td>6,74</td>
<td>1,00</td>
</tr>
<tr>
<td>2.0</td>
<td>3,04</td>
<td>7,51</td>
<td>1,11</td>
</tr>
<tr>
<td>4.0</td>
<td>3,43</td>
<td>8,47</td>
<td>1,26</td>
</tr>
<tr>
<td>6.0</td>
<td>3,94</td>
<td>9,73</td>
<td>1,44</td>
</tr>
<tr>
<td>8.0</td>
<td>4,63</td>
<td>11,41</td>
<td>1,69</td>
</tr>
<tr>
<td>10.0</td>
<td>5,60</td>
<td>13,81</td>
<td>2,05</td>
</tr>
</tbody>
</table>

Results for the long plate (\( \lambda a = 4.52 \))

Here the parameter \( k \) shows how many times the deflections and bending moments increase with increasing parameter \( \alpha \). This is clearly demonstrated in Fig. 4, where the corresponding graphs are presented.
4 Conclusions

1. It has been established that even a slight inhomogeneity of the base can cause the appearance of bending moments and transverse forces in the plate.

2. The conducted studies have shown that the most tangible effect is the inhomogeneity of the base for long plates, in which the values of deflections and bending moments increase by 1.5 - 2 times. For plates of medium length, the increase in deflections and bending moments can be 15-30%.

3. The method proposed in the article makes it possible to take into account the influence of the inhomogeneity of the base on the stress-strain state of the plate located on it, which is of undoubted practical importance.

References

1. Leontiev N.N. and other, Fundamentals of the theory of beams and plates on a deformable base, MSUCE, 1982 – 119 P.
2. Vlasov V.Z., Leontiev N.N. Beams, plates and shells on elastic foundation, Fizmatlit, 1960 – 491 P.
3. Leontiev N.N., Sobolev D.N., Variational principles of structural mechanics and basic theorems on elastic systems, MSUCE, 1980 – 54 P.

8. Leontiev A.N., Balandina K.V. Influence of Inhomogeneity of the Foundation on the Stress-strain State of the Design Located on it MATEC Web of Conferences Volume 196 (2018) XXVII R-S-P Seminar, Theoretical Foundation of Civil Engineering (27RSP) (TFCCE 2018) Rostov-on-Don, Russia, September 17-21, 2018, 01009 Published online: 03 September 2018 DOI: 10.1051/matecconf/201819601009


