Optimizing liquid fuel storage and distribution: simulation of outflows from vertical cylindrical vessels using Mathcad

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Abstract. In this article, the issues of fluid outflow from vertical cylindrical tanks through various bottom openings were investigated. The solution of the problem was reduced to the solution of the Cauchy problem, which was obtained analytically and numerically in the environment of the Mathcad package using built-in functions. The presented algorithms and Mathcad documents allow us to obtain a priori information about the hydrodynamic modes of emptying vessels, which increase the reliability, safety and durability of containers for storing petrochemical products.

1 Introduction

Differential equations are one of the most effective tools for the mathematical solution of practical problems. They are especially widely used in theoretical mechanics, physics and hydrodynamics [7-9].

The mathematical study of any problem concerning the real world is divided into three main stages [6]:

a) the construction of a mathematical model of the phenomenon;

b) studying this mathematical model and obtaining a solution to the corresponding mathematical problem;

c) the application of the results obtained to the practical issue from which the solution of this mathematical model arose, and the search for other issues to which it is applicable.

For example, some mathematical methods that arose for the first time in the study of hydrodynamics turned out to be applicable to aerodynamics, the theory of electric and magnetic fields, the theory of gravity and many other fields of physics and engineering.

When constructing a mathematical model of a phenomenon or process, its idealization and formalization are necessary. When idealizing a phenomenon, conditions that significantly affect it are separated from conditions that do not have a significant impact (at least in the approximation in which we solve the problem). For example, when studying the motion of a pendulum in the first approximation, air resistance, friction at the point of suspension, flexibility or, the shape of the load, etc. are neglected. This is how an idealized scheme of the phenomenon under study arises, called a mathematical pendulum in physics.

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Thus, many questions of physics, chemistry, economics, engineering and other fields of knowledge are reduced to the following task: to find a function f, having some equation, which, in addition to this function and the arguments on which it depends, also includes its derivatives up to and including some order. Such equations are called differential equations. If the desired function depends on only one argument, the equation is called an ordinary differential equation. Otherwise, it is called a partial differential equation. If the equation includes an nth-order derivative of the desired function, but does not include derivatives of higher orders, it is said that the order of this equation is equal to n.

Solving problems of physics or mechanics using differential equations falls into the following stages [6]:

a) drawing up a differential equation;
b) the solution of this equation;
c) research of the received solution.

The following sequence of actions is recommended:

1. To establish the quantities that change in a given phenomenon and to identify the physical laws connecting them.
2. Select the independent variable and the function of this variable that we want to find.
3. Based on the conditions of the problem, determine the initial or boundary conditions.
4. Express all the quantities appearing in the problem condition in terms of an independent variable, the desired function and its derivatives.
5. Based on the conditions of the problem and the physical law that this phenomenon obeys, make a differential equation.
6. Find a general solution or a general integral of a differential equation.
7. According to the initial or boundary conditions, find a particular solution.
8. Investigate the resulting solution.

In many cases, the formulation of a first-order differential equation is based on the so-called "linearity of the process in a small way", i.e., on the differentiability of functions expressing the dependence of quantities. As a rule, it can be assumed that all the quantities involved in a particular process change at a constant rate over a short period of time. This makes it possible to apply laws known from physics that describe uniformly occurring phenomena to compile a relationship between the values, i.e., the quantities involved in the process, and their increments. The resulting equality is only approximate, since the values change even in a short period of time, generally speaking, unevenly. But if you divide both parts of the resulting equality by and go to the limit when it tends to zero, you get an exact equality. It contains time t, physical quantities that change over time and their derivatives, i.e., it is a differential equation describing this phenomenon. The same equation in differential form can be obtained by replacing the increment with a differential, and the increment of functions with the corresponding differentials.

Thus, when composing a differential equation, a kind of "snapshot" of the process is taken at a given time, and when solving the equation using these snapshots, we restore the flow of the process. So, the basis for solving physical problems using differential equations is the general idea of linearization – replacing functions at small intervals of changing the argument with linear functions. And although there are processes (for example, Brownian motion) for which linearization is impossible because there is no rate of change of some quantities at a given time, in the vast majority of cases the method of differential equations works flawlessly.

The expansion of the range of industries in modern production has intensified research on the processes of liquid leakage from tanks through holes and nozzles. This has attracted increased attention to theoretical analysis, including safety issues for power and hydraulic engineering equipment, improving oil production processes, optimizing artificial irrigation, improving the reliability of water treatment systems, and other aspects.

Due to the complexity of carrying out experimental measurements of the hydrodynamic properties of most of these technological processes, it becomes necessary to develop analytical and numerical models to evaluate the various stages of fluid outflow processes.

In light of this, it becomes very important to comprehensively study the hydrodynamic state of reservoirs with holes that can be adjusted and those that cannot.

Reservoirs are the main storage sites for various liquids, including oil and petroleum products. Among the most important technological processes related to tanks, the operations of filling, storing and emptying them are distinguished. While filling the tank usually does not cause significant problems for hydraulics, emptying can be considered as a direct task of hydraulics.

2 Methods and tools

Consider the outflow of liquid from a vertical cylindrical tank. In the bottom of a vertical cylindrical vessel filled with liquid and having a height of H and a base radius of R, a small hole with an area of s is made (Fig. 1). A differential equation describing the process of liquid outflow from it was drawn up [1].

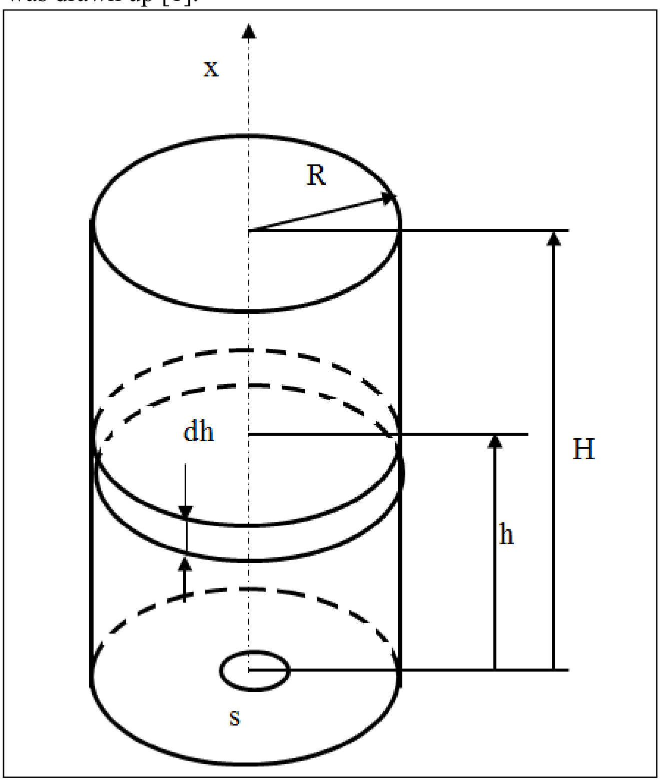

Fig. 1. Changing the liquid level in a cylindrical tank.

Observations show that at first the liquid flows out of the tank quickly, and as the water level decreases, the rate of its outflow decreases. Therefore, it is necessary to take into account the relationship between the flow rate v and the height h of the liquid column above the hole. Experiments conducted by the Italian physicist Torricelli have shown that the velocity v is approximately expressed by the formula

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$$v = k \cdot \sqrt{2 \cdot g \cdot h}$$

where g is the acceleration of gravity;

k is a dimensionless coefficient depending on the viscosity of the liquid and the shape of the hole (for example, for water in the case of a round hole, k = 0.6).

We investigate the expiration process over a period of time [t;t+Δt] . Let the height of the liquid above the hole be h at the beginning of this interval, and at the end of it it decreased and became

$$h + \Delta h$$

where Δh is the "increment" of height (which is obviously negative).

Then the volume of the liquid flowing out of the vessel is equal to the volume of the cylinder with a height |Δh|= -Δh and a base area of·R^2, i.e. ΔV=-π·R^2·Δh .

This liquid poured out in the form of a cylindrical trickle having a base area of s. Its height is equal to the path traveled by the liquid flowing out of the vessel over a period of time [t;t+Δt].At the beginning of this time interval, the expiration rate was equal according to Torricelli's law

$$k \cdot \sqrt{2 \cdot g \cdot h}$$

This means that the volume of liquid spilled over a period of time [t;t+Δt] is calculated using the formula

$$\Delta V = k \cdot \sqrt{2 \cdot g \cdot h} \cdot s \cdot \Delta t$$

We have obtained two expressions for the volume of liquid poured out of the vessel over a period of time [t;t+Δt].By equating these expressions, we obtain the equation

$$-\pi \cdot R^2 \cdot \Delta h = k \cdot \sqrt{2 \cdot g \cdot h} \cdot s \cdot \Delta t$$

Divide both parts of equation (1.1) by Δt and proceed to the limit at ΔT→0.

Because $\lim\limits_{\Delta t \to 0} \frac{\Delta h}{\Delta t} = h'$ then we get a first-order differential equation

$$-\pi \cdot R^2 \cdot h' = k \cdot \sqrt{2 \cdot g \cdot h} \cdot s \ .$$

Let us formulate the Cauchy problem of the velocity of liquid outflow from a vertical cylindrical vessel. From equation $-\pi \cdot R^2 \cdot h' = k \cdot \sqrt{2 \cdot g \cdot h} \cdot s$. we obtain the differential equation

$$\frac{dh}{dt} = -\frac{k \cdot s \cdot \sqrt{2 \cdot g \cdot h}}{\pi \cdot R^2} \ ,$$

under the initial condition

$$h_{t=0} = H_o,$$

where H_o – is the initial liquid level in the tank.

The differential equation $\frac{dh}{dt} = -\frac{k \cdot s \cdot \sqrt{2 \cdot g \cdot h}}{\pi \cdot R^2}$, can be solved analytically and numerically by specifying a different type of holes at the bottom of a vertical cylindrical tank. Consider the following types of bottom holes:

1. A round hole of radius r. In this case, the area s can be calculated using the well-known formula $s = \pi \cdot r^2$.
2. A square hole with side a. The area is found by the formula $s = \mathrm{a}^2$.
3. A rectangular hole with sides a and b. The expiration area was determined by the formula $s = a \cdot b$.
4. A triangle with sides a, b, c. The area was calculated using the Heron formula $s = \sqrt{p \cdot (p-a) \cdot (p-b) \cdot (p-c)}$, $p = \frac{a+b+c}{2}$, where p – semiperimeter.

Equation $\frac{dh}{dt} = -\frac{k \cdot s \cdot \sqrt{2 \cdot g \cdot h}}{\pi \cdot R^2}$, was integrated in the case when the bottom hole is a circle of radius r. In this case, the equation will be rewritten as

$$\frac{dh}{dt} = -\frac{k \cdot \pi \cdot r^2 \cdot \sqrt{2 \cdot g} \cdot \sqrt{h}}{\pi \cdot R^2} .$$

By shortening, separating variables and denoting

$$K = \frac{k \cdot r^2 \cdot \sqrt{2 \cdot g}}{R^2},$$

We get the equation

$$\frac{dh}{\sqrt{h}} = -K \cdot dt .$$

Integrate the right and left sides of the last equality

$$\int \frac{dh}{\sqrt{h}} = -K \cdot \int dt \quad \Rightarrow \quad \frac{h^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = -K \cdot t + C \quad \Rightarrow \quad 2 \cdot \sqrt{h} = -K \cdot t + C$$

$$\Rightarrow$$

$$\sqrt{h} = -\frac{K \cdot t}{2} + \frac{C}{2} \quad \Rightarrow \quad h(t) = \left(\frac{C}{2} - \frac{K \cdot t}{2}\right)^2$$

The constant of integration with was found using the initial condition $h_{t=0} = H_o$,. Substituting into the resulting general solution $t = 0$ и $h(t = 0) = H_o$, we will get

$$H_o = \left(\frac{C}{2} - \frac{K \cdot 0}{2}\right)^2 \Rightarrow H_o = \frac{C^2}{4} \Rightarrow C = 2 \cdot \sqrt{H_o}.$$

Then the partial solution of the differential equation $\frac{dh}{dt} = -\frac{k \cdot s \cdot \sqrt{2 \cdot g \cdot h}}{\pi \cdot R^2}$, has the form

$$h(t) = \left(\sqrt{H_o} - \frac{K \cdot t}{2}\right)^2 .$$

Using the last equality, it is possible to find the time of complete emptying of the tank. Assuming $h(t) = 0$, let 's solve the equation $\left(\sqrt{H_o} - \frac{K\cdot t}{2}\right)^2 = 0$

$$\sqrt{H_o} - \frac{K\cdot t}{2} = 0 \;\Rightarrow\; \frac{K\cdot t}{2} = \sqrt{H_o} \;\Rightarrow\; t = \frac{2\cdot\sqrt{H_o}}{K}\,.$$

Equation $\frac{dh}{dt} = -\frac{k\cdot s\cdot\sqrt{2\cdot g\cdot h}}{\pi\cdot R^2}$, is integrated similarly in cases 2-4, when the bottom holes through which the liquid flows are a square, rectangle and triangle, respectively.

The analytical solution of the problems of fluid outflow from cylindrical vessels is possible in the simplest linear case. In case of non-linearity, numerical methods for solving differential equations are used using mathematical packages, for example, the Mathcad package.

Mathcad is a computer–aided mathematical design system focused on the preparation of interactive documents with calculations, as well as text and visual support for mathematical calculations [2-5].

The built-in odesolve function is used for the numerical solution of ordinary differential equations and systems of differential equations in the Mathcad package environment.

Solution steps:
- entering the Given keyword to use the Mathcad solution block;
- we define the differential equation and its constraints using Boolean equality (Ctrl=). The differential equation can be written using operators of the type $\frac{d}{dt}$ $\quad and \quad$ $\frac{d^2}{dt^2}$ or in the form of x'(t) and x"(t). To set the initial approximation, values are entered for $x(t)$ and its first derivatives at the starting point at $t = 0$, in our case it is $x(0) = H_0$. Mathcad check the correctness of the type and number of constraints;
- we introduce the odesolve(t,b,step) function with the integration variable t and the numerical values of the endpoint b and the step step.

Thus, the above analytical solutions can be compared with the numerical ones found in the Mathcad package environment, with different values of the initial data.

3 Equations and mathematics

Mathcad is a computer–aided mathematical design system focused on the preparation of interactive documents with calculations, as well as text and visual support for mathematical calculations [16-19]. The Mathcad package (Math Soft Inc.) is quite common both in engineering practice and in the scientific environment. Its characteristic feature is the use of conventional mathematical notation, standard mathematical functions and formulas. Therefore, a Mathcad document on the monitor screen looks like an ordinary mathematical text from a monograph, textbook or lecture notes. The package is used mainly for numerical calculations, but has a built-in symbolic processor of the Maple computer algebra company MKM (Maple Kernel Mathsoft), which allows you to perform many analytical transformations, for example, direct and inverse Laplace and Fourier transforms. Currently, there is a trend towards convergence and integration of various mathematical packages. For example, the latest versions of the Mathematica and Maple packages have advanced features for visual programming, and the Mathcad and Matlab packages allow you to work together to solve the same problem [20]. At the same time, many numerical analysis problems are easily solved in the Mathcad package (solving systems of linear and nonlinear equations, solving ordinary differential equations and their systems, solving some partial differential equations, approximating functions, etc.) [21-24].

The built-in function rkfixed(y,x1,x2,m,D) is used for the numerical solution of ordinary differential equations and systems of differential equations in the Mathcad package environment. This function implements a numerical algorithm of the fourth-order Runge-Kutta method with a fixed step of splitting the integration segment. In addition, the Mathcad package contains a wide range of functions for the numerical solution of DU, which use the specific properties of a particular differential equation to ensure sufficient speed and accuracy when searching for a solution.

The rkfixed(y,x1,x2,m,D) function has five arguments, where

- • y is a column vector of initial conditions of size n, where n is the order of the differential equation or the number of equations in the system if a system of differential equations is being solved. For first-order equations, the column vector of initial values is a scalar (point) yo=y(x1).
- • x1,x2 are the boundary points of the integration segment on which a numerical solution of differential equations is sought. The initial conditions that are set in the vector column of the initial conditions y are the values of the desired solution at point x1.
- • m is the number of points, not counting the initial one, in which the desired approximate solution is sought (the number of intervals into which the integration segment is divided [x1, x2]). This argument defines the number of rows (m+1) in the decision matrix, which is calculated by the rkfixed function.
- • D(x,y) is a function, in the form of a vector column of n elements containing the first derivatives of unknown functions.
- The result of the solution is a matrix with two columns and m+1 row, while:
- • the first column of the matrix contains the argument points where the solution of our differential equation is determined;
- • the second column of the matrix contains the found values of the desired solution yi = y(xi) at the nodal points.
- To solve the first-order remote control system, it is necessary:
- • bring the system to its normal form, i.e. solve all equations with respect to the first derivative;
- • define a column vector y containing the initial values for each unknown function yi(x);
- • define the function D(x,y), in the form of an n-dimensional vector column that contains the first derivatives of each of the unknown functions.
- • refer to the rkfixed function.

The rkfixed function returns a matrix of solutions, whose first column will contain m+1 points where an approximate solution is being searched, and the remaining columns contain the values of the found approximate solutions yi at the corresponding points of the argument xi (i=0,1,2,...,m).

In addition to the rkfixed function, the Mathcad package contains the built-in Rkadapt function, which has the same arguments, but this built-in function implements a numerical algorithm for solving the Runge-Kutta method with a variable (adaptable) integration step h. At the same time, if the desired solution changes "slowly", then the package increases the value of h, reducing the number of split points, speeding up the calculation without losing the accuracy of the solution. Where the solution changes abruptly, the step h decreases. However, despite the fact that the numerical solution uses a variable step in calculations, the resulting solution (in the Z array) will contain m+1 points, i.e. the solution will be given on a uniform grid.

The numerical solution of a first-order differential equation using the built-in rkfixed function is shown in Figure 3.1.

To solve differential equations in the Mathcad package environment, the built-in odesolve function is also used.

The function has the following syntax:

- odesolve(t,b,step),
- where t is the integration variable;
- b is the end of the integration interval;
- step – step size (optional parameter).

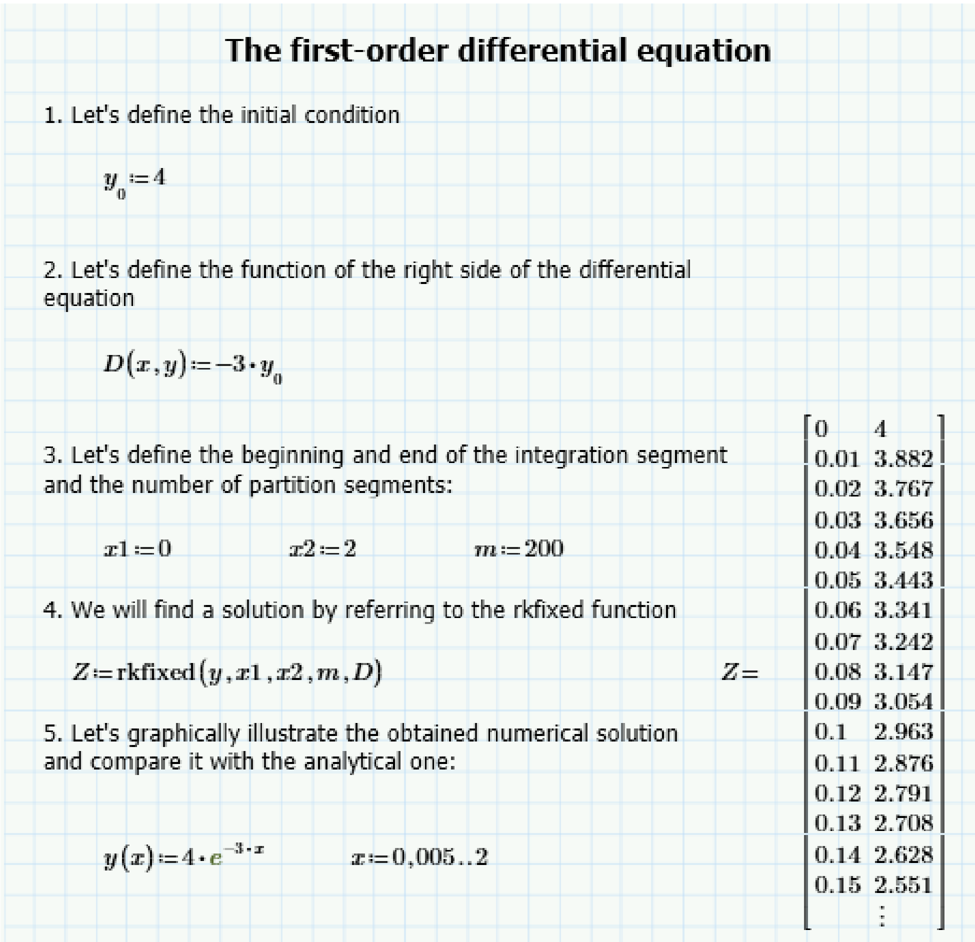

Fig. 2. Numerical solution of a first-order differential equation using the built-in rkfixed function.

In the environment of the Mathcad mathematical package, the following fluid flow problems have been numerically solved. A cylindrical tank with a diameter of R = 2 meters has a round hole with a radius of r = 0.09 m at the bottom. The initial liquid level in the tank is H0 = 3 meters. It is required to establish the dependence of the liquid level in the tank on time t. For water, kerosene, gasoline, the coefficient of the expiration rate is assumed to be k = 0.62.

To solve the formulated Cauchy problem of fluid outflow from a cylindrical vessel, the differential equation $\frac{dh}{dt} = -\frac{k \cdot s \cdot \sqrt{2 \cdot g \cdot h}}{\pi \cdot R^2}$, was obtained under the initial condition $h_{t=0} = H_o$,.

The numerical solution found using the odesolve function can be compared with the analytical solution (see Fig. 3).

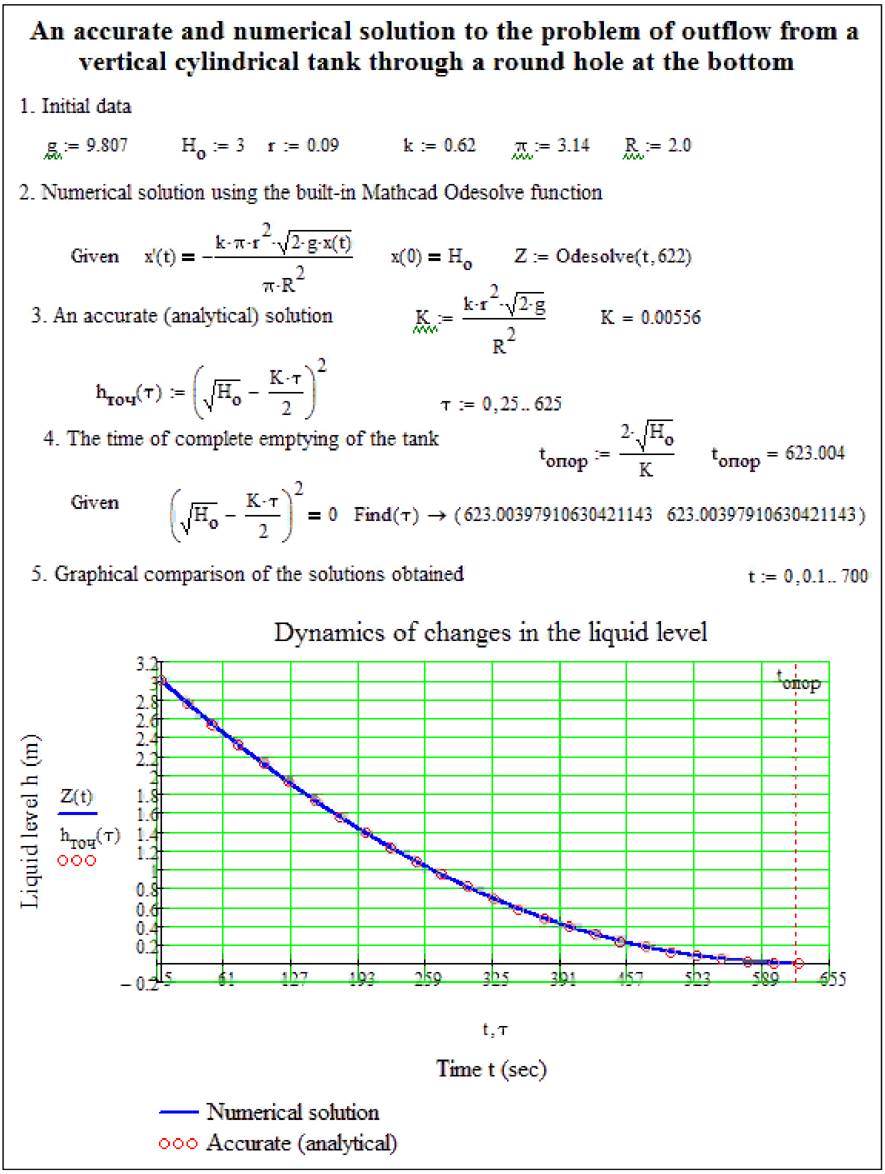

Fig. 3. Comparison of numerical and analytical solutions.

The results of modeling the dependence of the liquid level in cylindrical tanks with a height of H0 = 3 meters and a radius of 2, 2.2 and 2.5 meters are shown in Figure 3. From the graphs of the solution, it can be seen that with an increase in the radius of the cylindrical tank, and, consequently, the volume of the liquid in it, the expiration time also increases.

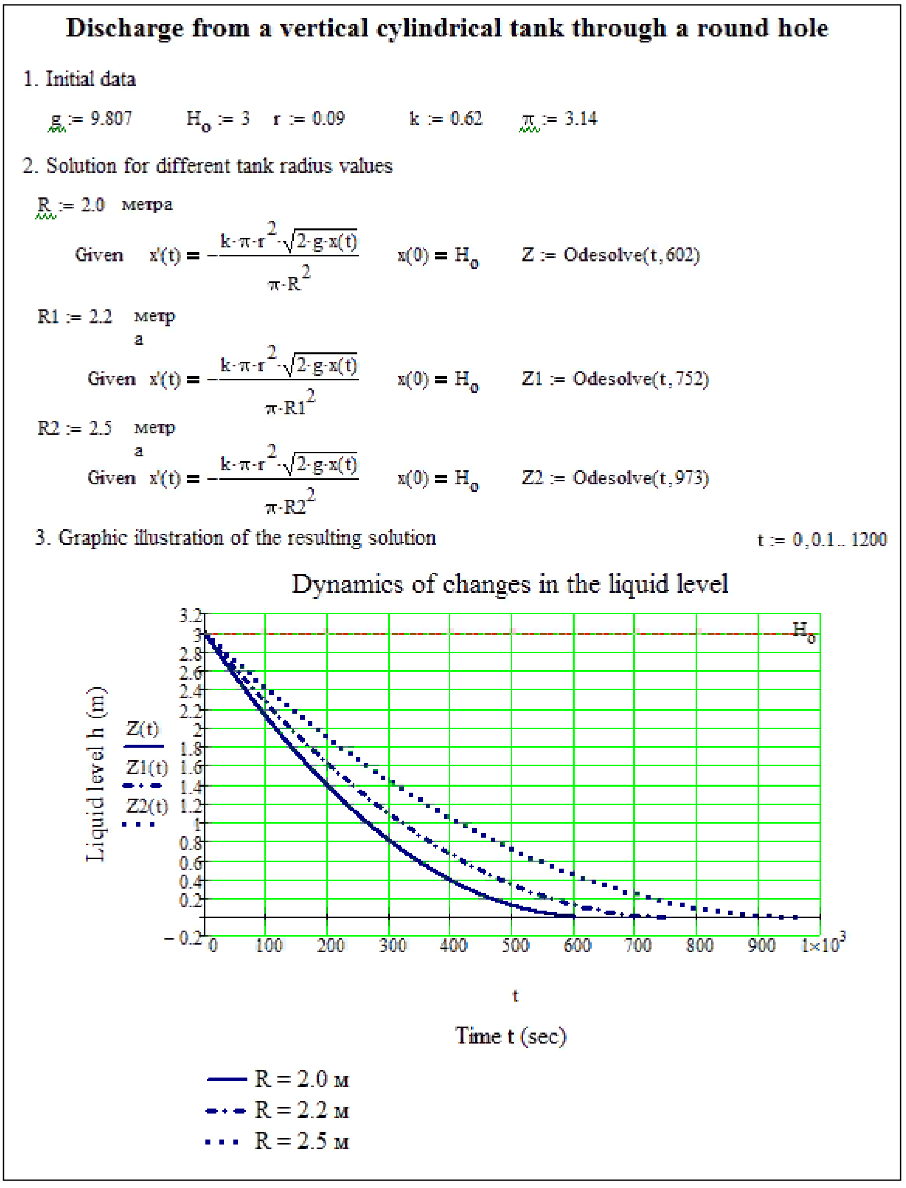

Fig. 4. Dynamics of fluid outflow from vertical cylindrical tanks of various radii.

Figure 4 shows the results of modeling the process of liquid outflow from reservoirs of equal volume at different radii of the drain opening r. From the graphs shown, it can be seen that with increasing radius, the emptying time decreases.

Discharge from a vertical cylindrical tank through a square hole

1. Initial data

$g := 9.807$ $H_o := 3$ $a := 0.1$ $k := 0.62$ $\pi := 3.14$

2. Solution for different tank radius values

$R := 2.0$ meters

Given $x'(t) = -\dfrac{k \cdot a^2 \cdot \sqrt{2 \cdot g \cdot x(t)}}{\pi \cdot R^2}$ $x(0) = H_o$ $Z := \mathrm{Odesolve}(t, 1450)$

$R1 := 2.2$ meters

Given $x'(t) = -\dfrac{k \cdot a^2 \cdot \sqrt{2 \cdot g \cdot x(t)}}{\pi \cdot R1^2}$ $x(0) = H_o$ $Z1 := \mathrm{Odesolve}(t, 1752)$

$R2 := 2.5$ meters

Given $x'(t) = -\dfrac{k \cdot a^2 \cdot \sqrt{2 \cdot g \cdot x(t)}}{\pi \cdot R2^2}$ $x(0) = H_o$ $Z2 := \mathrm{Odesolve}(t, 2250)$

3. Graphic illustration of the resulting solution $t := 0, 0.1 .. 2400$

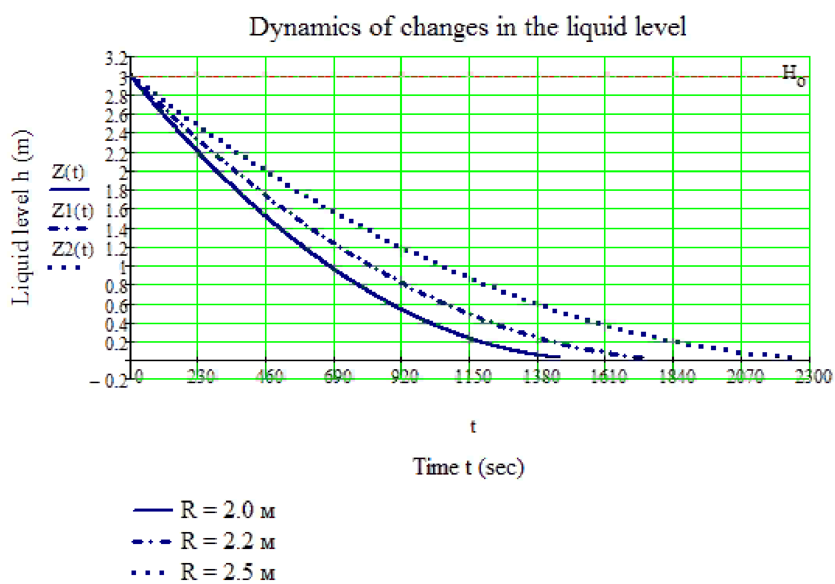

Fig. 5. Modeling in the environment of the Mathcad package of the process of liquid outflow at different radii of the drain hole.

The results of calculating the process of liquid outflow from cylindrical vessels of various radii through a square drain hole are shown in Figure 5, through a rectangular hole with sides a and b – in Figure 6, and through a triangular bottom hole – in Figure 7.

Discharge from a vertical cylindrical tank through a rectangular opening

1. Initial data

$g := 9.807 \quad H_o := 3 \quad a := 0.06 \quad b := 0.17 \quad s := a \cdot b \quad k := 0.62 \quad \pi := 3.14$

2. Solution for different tank radius values

$R := 2.0$ meters

Given $x'(t) = -\frac{k \cdot s \cdot \sqrt{2 \cdot g \cdot x(t)}}{\pi \cdot R^2} \quad x(0) = H_o \quad Z := \text{Odesolve}(t, 1450)$

$R1 := 2.2$ meters

Given $x'(t) = -\frac{k \cdot s \cdot \sqrt{2 \cdot g \cdot x(t)}}{\pi \cdot R1^2} \quad x(0) = H_o \quad Z1 := \text{Odesolve}(t, 1752)$

$R2 := 2.5$ meters

Given $x'(t) = -\frac{k \cdot s \cdot \sqrt{2 \cdot g \cdot x(t)}}{\pi \cdot R2^2} \quad x(0) = H_o \quad Z2 := \text{Odesolve}(t, 2250)$

3. Graphic illustration of the resulting solution $\quad t := 0, 0.1 .. 2400$

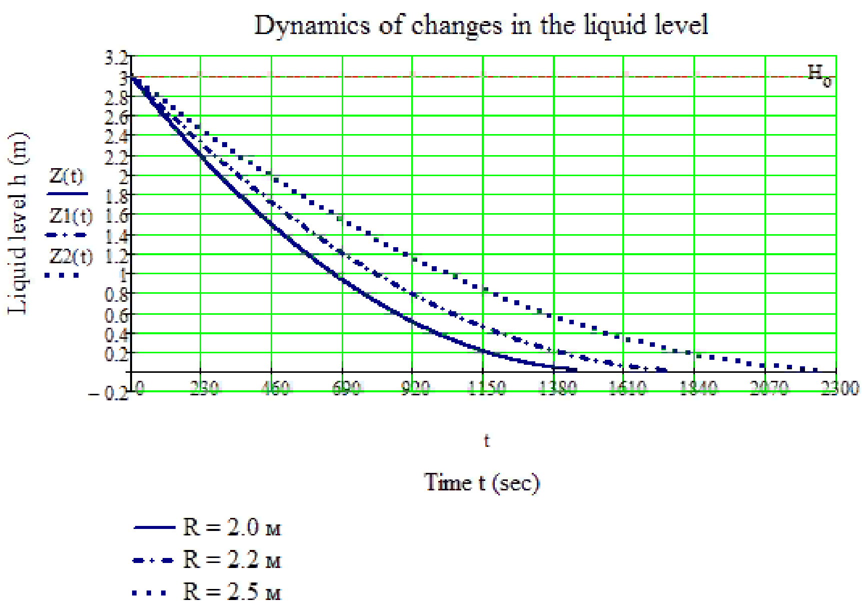

Fig. 6. Simulation of the process of liquid outflow from vertical cylindrical tanks of various radii through a rectangular drain hole.

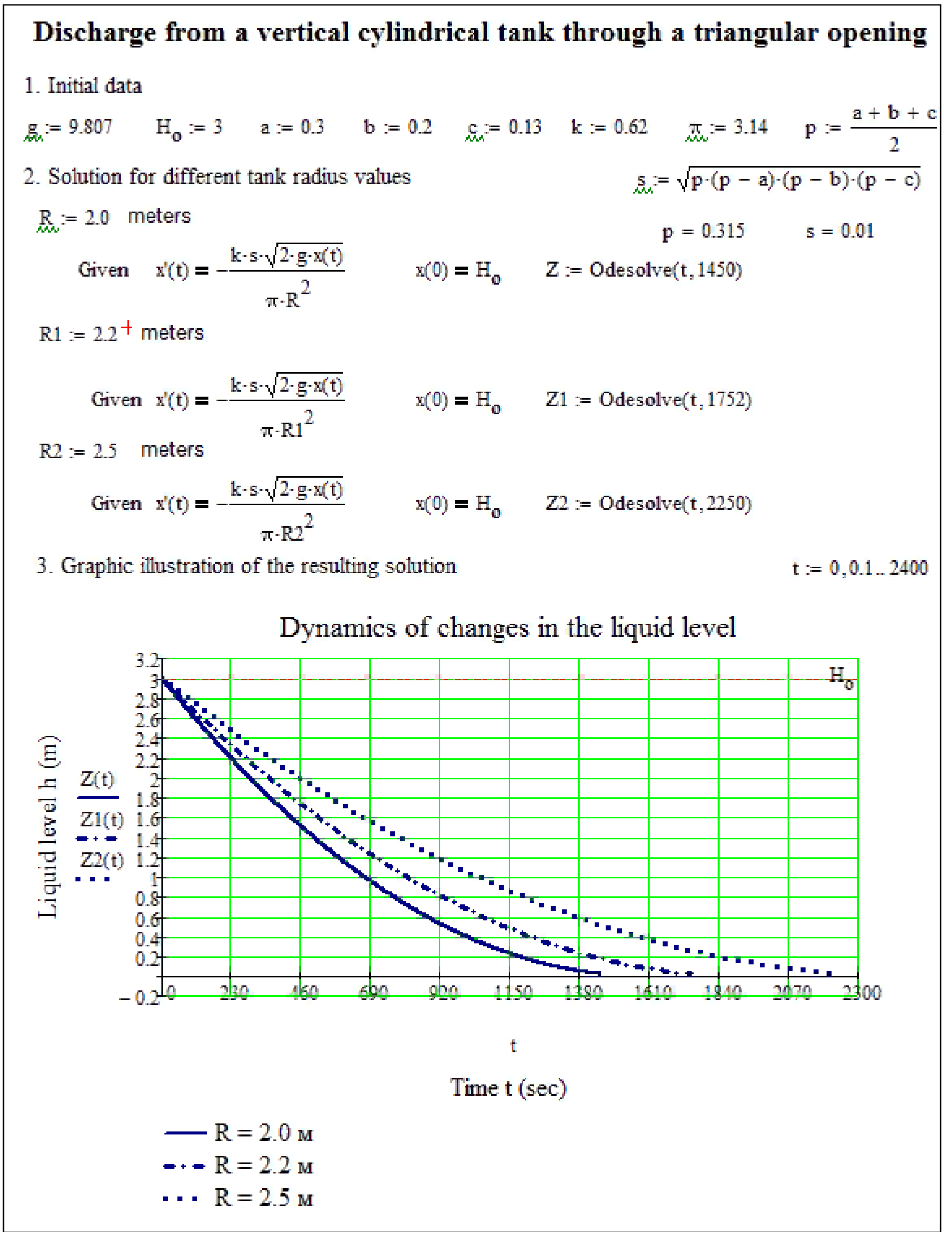

Fig. 7. Simulation of the process of liquid outflow from vertical cylindrical tanks of various radii through a triangular drain hole.

The obtained analytical and numerical results of solving the problem can serve as the basis for obtaining effective a priori information on the distribution of hydrodynamic characteristics of vertical cylindrical tanks. The latter allows, knowing the laws of movement of the liquid level and the flow rate, to choose technological modes that increase the reliability, safety and durability of tanks.

4 Conclusion

Thus, the article considers the issues of fluid outflow from vertical cylindrical tanks of various capacities with a given geometry of bottom holes. Dynamic expiration modes and emptying time of containers are determined. The formulated Cauchy problems are solved analytically (for the simplest cases) and numerically, in the environment of the Mathcad package using built-in functions. The results obtained allow us to evaluate the dynamics and time of the outflow of liquids from reservoirs, increasing the reliability and safety of processes and facilities in the oil and gas industry.

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