Entropy and heat capacity of a degenerate Fermi gas in a three-phase system with different spatial dimensions

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Abstract. This work is a generalization and further development of the previously proposed theory of spatial transitions of atoms in configurations with the number of spatial dimensions D = 1, 2, 3. Using analytical and numerical methods were obtained relations linking the probabilities of transitions $1 \leftrightarrow 3$ and $2 \leftrightarrow 3$ and the average number of atoms in configurations 1, 2 and 3 in the assumption of the existence of only these pairs of spatial configurations, as observed in previously conducted experiments. Also, a similar approach was implemented in the situation with the possible simultaneous existence of all three spatial configurations with the same transitions plus transitions $1 \leftrightarrow 2$, and also found and the average number of atoms, although the corresponding experiments have not yet been carried out. In the present paper a three-phase system of degenerate fermi-gas in states with spatial dimensions D=1, 2, 3 with possibility of probabilistic transitions from one phase to another is considered. The calculated values of entropy and heat capacity in these phases are significantly different. These phase transitions with a jump change in the heat capacity are thus phase transitions of the 2nd kind.
Keywords: Fermi Gas, Spatial Dimension, Heat Capacity, Phase, Phase Transitions.

1 Introduction

The study of the properties of a low-dimensional Fermi gas is an urgent task, since without such information, progress in the preparation and study of Josephson junctions, in which the weak coupling is realized through a two-dimensional electron gas, is impossible. These systems are prepared using semiconductor heterostructures with superconducting contacts [1].

The theory of a degenerate Fermi gas in the space *D*=3 with the calculation of its thermodynamic potentials, in particular, Ω, F - potential, as well as entropy and heat capacity, is described in the monograph [2].

On the other hand, in the work of one of the authors [3], using the example of a hydrogen-like atom, the possibility of mutual transformations of quantum systems with a change in their spatial dimension was shown $D_{1,2,3} \equiv 1,2,3$ (Section 2 of this work). The

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result of this work confirms the provisions of the general theory of probabilistic transformations of such systems [4, 5].

Hence it follows that the mechanism of formation of matter in subspaces $D_{1,2}$ is possible, based in the general case on the representation of the wave function $\psi_3 \equiv \psi_3\left(\{x_i, y_i, z_i\}\right)$, which is a function of the sets of coordinates $\{x_i, y_i, z_i\}$ of all particles of the quantum system in D_3. Such a representation can be carried out by expanding in combination $\psi_{1,2} \equiv \psi_1\psi_2$ of wave functions ψ_1 and ψ_2 the same system in D_1 and in D_2, respectively:

$$\psi_3 = \sum_{1,2} C_{3;1,2}\, \psi_{1,2}\,, \tag{1}$$

$$\psi_1 \equiv \psi_1\left(\{z_i\}\right), \quad \psi_2 \equiv \psi_2\left(\{x_i, y_i\}\right),$$

where the summation in Eq. (1) is performed over the sets of quantum numbers of the system in $D_{1,2}$, and the expansion coefficients $C_{3;1,2}$ can be obtained using the relation between the volume elements of the configuration spaces[†]

$$dV_3 = dV_1\, dV_2$$
$$dV_3 = \prod_i dx_i\, dy_i\, d\,z_i, \quad dV_1 = \prod_i d\,z_i, \quad dV_2 = \prod_i dx_i\, dy_i\,. \tag{2}$$

Expansion coefficients are equal

$$C_{3;1,2} = \int \psi_{12}^{*}\, \psi_3\, dV_3\,. \tag{3}$$

In this case, the coordinate variables in $\psi_{1,2}$ must be expressed as in ψ_3, through the variables in D_3.

Then, as was shown in [3] (see also [7] and [8]), the probability $W_{3;1,2}$ of finding a system originally located in D_3, in the subspace D_1 (or in D_2) is equal to

$$W_{3;1,2} = \left|C_{3;1,2}\right|^2. \tag{4}$$

Similarly, from the representation ψ_2, which is a function of the sets of coordinates $\{x_i, y_i\}$, by means of the functions ψ_{1x} and ψ_{1y} as functions of the sets $\{x_i\}$, $\{y_i\}$ respectively, follows expansion

$$\psi_2 = \sum_{1x,1y} C_{2;1x,1y}\, \psi_{1,1}, \quad \psi_{1,1} \equiv \psi_{1x}\, \psi_{1y}\,, \tag{5}$$

where

$$C_{2;1x,1y} = \int \psi_{11}^{*}\, \psi_2\, dV_2\,. \tag{6}$$

The probability of detection in D_1 the system originally located in D_2 is

$$W_{2;1x,1y} = \left|C_{2;1x,1y}\right|^2. \tag{7}$$

These relations take place in the region of the discrete spectrum, when the system is localized in limited regions of spaces $D_{1,2,3}$ with the corresponding sets of quantum numbers. For example, for the system of fermions considered in this work, this is true for atoms with a total spin $1/2$ or for nucleons with their three-quark structure.

†The terminology of the book [6] is used here.

It should be noted that some other atoms were indeed obtained in $D_{1,2}$ the experiments of the authors of [9–12].

Thus, from these positions, a three-phase system of fermions with different spatial dimensions $D_{1,2,3}$ in each phase can exist. In this case, according to Eqs. (4), (7) of work [3], probabilistic transitions from one phase to another are possible with a change in heat capacity C_{CP} and entropy S_{EN}. As will be seen below, the heat capacity in such transitions changes abruptly. According to the terminology of [2], these transitions are referred to as second-order phase transitions.

In this paper, we calculate the heat capacity and entropy of these spatial phases with accompanying results (Fermi energy, etc.).

2 Basic considerations

The number of fermion particles dN in an element of the phase volume $d\bar{p}$ in spaces $D_{1,2}$, by analogy with Eq. (56.3) of the monograph [2] in D_3 (as well as further and other general relations that follow from it) is given by the following expressions:

1) in D_1 with the designation of the "one-dimensional" coordinate by x

$$dN = \frac{gL}{2\pi\hbar} \frac{dp_x}{e^{(\varepsilon-\mu)/T}+1}. \tag{8}$$

Here, $g = 2s+1$ is the number of fermion spin states, μ is the chemical potential, and L is the linear "volume" of the system.

As in [2], we proceed further to integration over $\varepsilon = p_x^2/2m$. Given the symbolic equality

$$\int_{-\infty}^{\infty} dp_x = 2\int_0^{\infty} dp_x$$

we obtain for the "linear concentration" C_C of fermions:

$$C_C \equiv \frac{N}{L} = \frac{g\sqrt{m}}{\sqrt{2}\pi\hbar}\int_0^{\infty}\frac{d\varepsilon/\sqrt{\varepsilon}}{e^{(\varepsilon-\mu)/T}+1}. \tag{9}$$

After changing the variable $\varepsilon/T = \mathrm{z}$, we arrive at the expression

$$C_C = \frac{g\sqrt{mT}}{\sqrt{2}\pi\hbar}\int_0^{\infty}\frac{dz/\sqrt{z}}{e^{z-(\mu/T)}+1}, \tag{9a}$$

which implicitly specifies the dependence of μ on T and C_C similarly to Eq. (56.5) [2].

Further, for the Ω-potential [1], using Eq. (8), we have:

$$\Omega = -\frac{LgT\sqrt{m}}{\sqrt{2}\pi\hbar}\int_0^{\infty}\frac{d\varepsilon}{\sqrt{\varepsilon}}\ln[1+e^{(\mu-\varepsilon)/T}]. \tag{10}$$

Integration by parts gives the expression

$$\Omega = -\frac{gL\sqrt{2m}}{\pi\hbar}\int_0^{\infty}\frac{d\varepsilon\,\sqrt{\varepsilon}}{e^{(\varepsilon-\mu)/T}+1}, \tag{10a}$$

similar to Eq. (56.6) [2].

The expression for the energy E of the system is obtained taking into account Eq. (8) by adding the factor ε in the integrand of the form Eq. (9) and multiplying by L. It is easy to see that we will then have $E = -\Omega/2$. On the other, hand $\Omega = -PV$ similar to the formula for the Ω-potential in D_3 [2], the corresponding formula in D_1 has the form $\Omega = -fL$. Here $f \equiv P_1$ is the force acting "from the side" of the entire system on the boundary point of the "one-dimensional chain" of fermions.

Thus, similar to Eq. (56.8) [2] in D_3 with the same physical meaning in D_1 formula has the form:

$$fL = 2E\,. \tag{11}$$

The value of the Fermi energy $\varepsilon_F = \mu|_{T=0}$ can be obtained from Eq. (9) by an obvious transformation of the integrand

$$C_C = \frac{g\sqrt{m}}{\sqrt{2}\pi\hbar}\int_0^{\varepsilon_F}\frac{d\varepsilon}{\sqrt{\varepsilon}}\,.$$

Performing elementary integration, we find:

$$\varepsilon_F = \frac{\pi^2\hbar^2 C_C^2}{2g^2 m}\,. \tag{12}$$

2) Further, similarly to item 1), we have in D_2 with the designation of "two-dimensional" Cartesian coordinates by x, y:

$$dN = \frac{gS}{(2\pi\hbar)^2}\frac{dp_x dp_y}{e^{(\varepsilon-\mu)/T}+1}\,, \tag{13}$$

where S - "surface volume" of the system.

Passing to polar coordinates in the two-dimensional phase space with the replacement $dp_x dp_y \to 2\pi p\,dp$, we find, taking into account the values of $\varepsilon = p^2/2m$, $d\varepsilon = p\,dp/m$ an expression similar to Eq. (9) for the "surface concentration":

$$C_C \equiv \frac{N}{S} = \frac{gm}{2\pi\hbar^2}\int_0^\infty \frac{d\varepsilon}{e^{(\varepsilon-\mu)/T}+1}\,. \tag{14}$$

In this case, an expression similar to Eq. (9a), which implicitly determines the dependence $\mu(T, C_C)$, is as follows:

$$C_C = \frac{gmT}{2\pi\hbar^2}\int_0^\infty \frac{dz}{e^{z-(\mu/T)}+1}\,. \tag{14a}$$

Similar to Eq.(10a), the expression for the Ω-potential in D_2 obtained in the same way has the form:

$$\Omega = -\frac{gSm}{2\pi\hbar^2}\int_0^\infty \frac{\varepsilon\,d\varepsilon}{e^{(\varepsilon-\mu)/T}+1}\,. \tag{15}$$

Adding a multiplier ε on the right side of Eq. (13) (see also the integrand Eq. (14)), we come from comparison with Eq. (15) to an equality $E = -\Omega$ with the corresponding modification of formula Eq. (11):

$$PS = E \tag{16}$$

and the definition of "pressure" $P \to P_2$ in D_2 as the force applied to the unit length of the curve, limiting the "two-dimensional phase".

The value of the Fermi energy is obtained from Eq. (14), which in D_2 at $T = 0$ has the form:

$$C_C = \frac{gm}{2\pi\hbar^2}\int_0^{\varepsilon_F} d\varepsilon\,.$$

From here we find

$$\varepsilon_F = \frac{2\pi C_C \hbar^2}{gm}. \tag{17}$$

3 Entropy and heat capacity of a degenerate Fermi gas in D$_{1,2}$

The heat capacities at constant volume $C_{V;1,2}$ and pressure $C_{P;1,2}$ in $D_{1,2}$ can be obtained by the method used in §58 of the monograph [2] by expanding the integral appearing in Eqs. (10a) and (15) of the form

$$I = \int_0^\infty \frac{f(\varepsilon)\, d\varepsilon}{e^{(\varepsilon-\mu)/T} + 1}$$

in powers of a small parameter - temperature, in the case of a degenerate Fermi gas. This expansion can be written in the form (see [2] - § 58):

$$I = \int_0^{\varepsilon_F} f(\varepsilon)\, d\varepsilon + \frac{\pi^2}{6} f'(\varepsilon_F) T^2 + \ldots \tag{18}$$

Taking into account the form $f(\varepsilon) = \sqrt{\varepsilon}$ in D_1 (see Eq. (10a)), we obtain in view of Eq. (12) and the identity of the Ω-potential and free energy F [2] for the values of the degeneracy temperature $T \to 0$:

$$F = F_0 - \frac{1}{6} \frac{g^2 Lm}{\hbar^2 C_C} T^2. \tag{18a}$$

Here F_0 is the free energy at $T = 0$, whose explicit form is of no importance for our purposes.

The entropy S_1 in D_1 is obtained from Eq. (18a) using the usual thermodynamic relation $S = -\partial F / \partial T$ with the result

$$S_1 = \frac{1}{3} \frac{g^2 Lm}{\hbar^2 C_C} T. \tag{19}$$

With the transition to dimensionless quantities ($\lambda_C = \hbar / mc$ - Compton fermion wavelength)

$$\tilde{L} = \frac{L}{\lambda_C}, \quad \tilde{C}_C = C_C \lambda_C, \quad \tilde{T} = \frac{T}{mc^2}, \tag{20}$$

and, taking into account the dimensionlessness in our units of the entropy itself, we obtain the value of the entropy of the degenerate Fermi gas in D_1:

$$S_1 = \frac{1}{3} \frac{g^2 \tilde{L} \tilde{T}}{\tilde{C}_C}. \tag{21}$$

Similar relations in D_2 have the form:

$$F = F_0 - \frac{1}{12} \frac{\pi g S m}{\hbar^2} T^2, \tag{22}$$

$$S_2 = \frac{\pi g \tilde{S} \tilde{T}}{6}, \tag{23}$$

with additional notation $\tilde{S} = S / \lambda_C^2$, moreover, S_2 Eq. (23), in contrast to S_1 Eq. (21), does not depend on the concentration C_C at all.

The heat capacity values $C_{V,\,P;\,1,\,2}$ in $D_{1,2}$, as in D_3 [2], are equal to the entropy $S_{1,\,2}$. This is due to the linear dependence of $S_{1,\,2}$ on T (see Eqs. (21), (23)) and is a consequence of the following relations

$$C_{V,\,P;\,1,\,2} = T\frac{dS_{1,\,2}}{dT} = \tilde{T}\frac{dS_{1,\,2}}{d\tilde{T}} = S_{1,\,2}\,,$$

moreover, as in D_3 [2], the heat capacities at constant volume and pressure are the same:

$$C_{V;\,1,\,2} = C_{P;\,1,\,2} \equiv C_{V,\,P;\,1,\,2}\,.$$

Thus,

$$C_{V,\,P;\,1} = \frac{1}{3}\frac{g^2\tilde{L}\tilde{T}}{\tilde{C}_C}\,, \tag{24a}$$

$$C_{V,\,P;\,2} = \frac{\pi g\tilde{S}\tilde{T}}{6}\,. \tag{24b}$$

Note that when using nonrelativistic atomic units of measurement of length and energy [6], expressed in this case in terms of the fermion mass m independent of D and e - elementary charge in $D = 3$, i.e.

$$\lambda_C \to r_B = \frac{\hbar^2}{me^2}, \quad mc^2 \to E_a = \frac{me^4}{\hbar^2}\,, \tag{25}$$

the results of the Eqs. (21), (23) and (24a, b) in dimensionless quantities are the same.

4 Conclusion

As can be seen from Eqs. (24a, b), the heat capacities in $D_{1,2}$, as well as in D_3 [2] are significantly different, i.e. when the dimension changes, the spaces change abruptly, and the considered transitions between phases with different spatial dimensions refer to phase transitions of the second kind [2]. In this case, the phases in $D_{1,2}$ will be formed from the phase in D_3, for example, in a parallelepiped with sides a, b, c.

Namely, the phase in D_2 is formed in the plane (a,b) parallel to the face of the parallelepiped (under the condition $a,b >> c$), and the phase in D_1 - on the line parallel to the edge "c" (under the condition $c >> a,b$). This follows from experiments, for example, by the authors of [9], in which the appearance of different spatial phases is due to the configuration of the experimental setup.

It should also be noted that measuring the heat capacity of a degenerate Fermi gas in $D_{1,2}$ in order to verify the obtained heat capacity values is a rather difficult experimental problem, because imparting a finite amount of heat to a degenerate system of fermions can bring it out of the state of degeneracy. This means that the amount of heat mentioned must be small enough, and therefore the change in temperature is also small. The experiments of the authors of [9–12] were also carried out at temperatures near absolute zero, but they recorded only the very fact of the appearance of atoms in $D_{1,2}$ and did not set the task of determining the temperature. Thus, the verification of our results is a more difficult problem than the one that was solved by the authors of [9–12], although in principle such an experiment can be set up.

Note also that the applicability of the Fermi-Dirac distribution in $D_{1,2}$ is not at all obvious. In this regard, setting up an experiment to detect the second-order phase transitions discussed in this work with verification of the results of the theory is even more desirable. Confirmation of our results would mean the applicability of the Fermi-Dirac distribution in this case as well.

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