Robust and Arbitrary Pole Placement using Static Output Feedback

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Abstract. This paper presents the output feedback pole assignment methods that attempt to achieve a robust solution where the arbitrary choose of the poles to be assigned are as insensitive as possible to perturbations in the closed loop system. The robust and arbitrary pole placement problem is formulated as a nonlinear and non-convex programming problem with a nonlinear constraint on matrix functions as well as a linear constraint. The numerical treatment is the subject of a comparative study of the proposed methods.

1 Introduction

Most proposed methods in the litterature of robust pole placement by static output feedback cannot produce the exact pole placement by their sensitivity, see the paper [12], where several literatures on the design of static output feedback for linear time invariant systems have been reviewed. In real plants, the dynamic of the system is not exactly known and can be subject to disturbances. Therefore, the best method for pole placement needs to consider the sensitivity of the located poles to possible errors in the system model or to external disturbances. The sensitivity to error is measured by the condition number. Therefore, the small variation of eigenvalues due to possible perturbations is guaranteed by a minimal condition number of the eigenvector matrix. Thus, robust pole placement refers to the ability to maintain the exact placement of desired poles in a closed loop in the presence of external perturbations and uncertainties by minimizing the condition number. The flexibility and importance of the method developed by [1] for solving the robust exact pole placement problem with state feedback for linear invariant systems leads us to apply the same approach followed at the beginning of development of the work for the case of pole placement problem with static output feedback. This latter approach is known as the famous mathematical Moore-Penrose pseudo-inverse approach applied to the solution of arbitrary linear systems.

This paper treats two methods to solve exact pole placement problem with any arbitrary desired poles (distinct, multiple or overlapping with the open-loop spectra) with minimization of the condition number. The utility of the proposed techniques lies in achieving exact placement of arbitrary poles, robustness and take into account other desired performance constraints. The developed procedure leads to a nonlinear, non-convex optimization problems with a nonlinear constraint on the matrix functions and a linear constraints. The numerical resolution of the proposed optimization problems is done using the *fmincon* function of Matlab

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The remainder of this paper is organized as follows. Section 2 describes a new formulation of robust pole placement problem using static output feedback control with an arbitrary choose of the pole to be assigned. Section 3 shows the technique used to relax the proposed objective function in order to obtain satisfactory numerical results. Finally, section 4 deals with a numerical results of the proposed methods with a comparative study.

Notations

The identity matrix is denoted by $\mathbb{I}$. $\sigma(S)$, S^+, S^T note the eigenvalues or spectra, the pseudo inverse and the transpose of matrix S respectively. $trace(S)$ is the trace of the matrix S

2 Robust and Arbitrary Pole Placement problem with Static Output Feedback: RAPP-SOF

Consider the following linear continuous-time invariant system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{1}$$

Where the vector elements are: sate vector $x \in \mathbb{R}^n$, the input vector $u \in \mathbb{R}^m$ and the output vector $y \in \mathbb{R}^p$.
The matrix components are: $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$, (A, B) is controllable and (A, C) is observable.
Consider the static output feedback (SOF) control law:

$$u(t) = Fy(t),\ F \in \mathbb{R}^{m\times p}, \tag{2}$$

to place a desired poles $\lambda_1, \ldots, \lambda_n$. Thus the closed-loop system is given in the following form :

$$\dot{x}(t) = (A + BFC)x(t), \tag{3}$$

Our goal is to compute a gain matrix F of the control law (2) such that the arbitrarily selected poles $\sigma(A + BFC) = \{\lambda_1, \ldots, \lambda_n\}$ can be precisely placed.
The closed-loop matrix (3) has the following real Jordan decomposition:

$$A + BFC = X\Sigma X^{-1} \tag{4}$$

Where Σ is the real Jordan matrix associated with the desired eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ and X is the eigenvector matrix.
An arbitrary set of poles can be placed by solving the equation (4) after having constructed the real Jordan matrix Σ.
Taking this change of variables $K = FC$, we have

$$BK = X\Sigma X^{-1} - A \tag{5}$$

which has for solutions

$$K = B^{+}(X\Sigma X^{-1} - A) - (B^{+}B - \mathbb{I})z, \tag{6}$$

where z is an arbitrary matrix. If and only if this first condition is satisfied:

$$(X\Sigma - AX)(\mathbb{I} - BB^{+}) = 0 \tag{7}$$

This is a direct application of the famous following lemma, which is nothing more than the Moore-Penrose pseudoinverse approach applied to the solution of arbitrary linear systems.

Lemma 1 *[10] [11]*
All solution S of the matrix system $MS = N$, $M \in \mathbb{R}^{p\times q}$, $N \in \mathbb{R}^{p}$ are given by

$$S = M^{+}N + (\mathbb{I} - M^{+}M)z,$$

where z is an arbitrary matrix. if and only if

$$MM^{+}N = N \tag{8}$$

Form (6) with the previous change of variable $K = FC$ we get this equation:

$$FC = B^{+}(X\Sigma X^{-1} - A),$$

and by applying lemma (1), we obtain the expression of the matrix gain solution to arbitrary pole placement problem.

$$F = B^{+}(X\Sigma X^{-1} - A)C^{+} \tag{9}$$

if and only if this second condition is satisfied:

$$B^{+}(A - X\Sigma X^{-1})(C^{+}C - \mathbb{I}) = 0. \tag{10}$$

The measure of the sensitivity of the eigenvalues of the closed-loop system is the condition number, which gives a bound on the relative error in the solution when a small perturbation is introduced.

$$Cond_F(X) = \|X\|_{fro} \left\|X^{-1}\right\|_{fro}, \tag{11}$$

where the norm: $\|.\|_{fro}$ is the Frobenuis norm

$$\|X\|_{fro}^{2} = \sum_{i,j=1}^{n} \left|X_{i,j}\right|^{2} = trace(XX^{T})$$

By looking for a well-conditioned matrix X that satisfies (7) and (10), we can make an accurate numerical calculation of the gain matrix F given by equation (9). According to the previous considerations, we are now in a position to formulate the problem of the robust and arbitrary pole placement using static output feedback as the following result.

Theorem 2 *The solution of the Robust and Arbitrary Pole Placement problem with static output feedback control of linear systems is given by the following optimization problem: Find X optimal solution to:*

$$(RAPP - SOF) \quad \begin{cases} \min \ trace(XX^{T})\ trace(X^{-1}X^{-T}) \\ subject\ to \\ \quad (\mathbb{I} - BB^{+})(AX - X\Sigma) = 0 \\ \quad B^{+}(A - X\Sigma X^{-1})(C^{+}C - \mathbb{I}) = 0 \end{cases} \tag{12}$$

Proof. The proof follows directly from the results above. It is sufficient to minimise the condition number (11) under the constraints of the equations (7) and (10) to have a robust and arbitrary pole placement with static output feedback. ■
In fact, the objective function is highly nonlinear and nonconvex. Moreover, the nonlinearity of the second condition in the variable X makes the optimization problem (RAPP-SOF) difficult to solve numerically. Subsequently, we provide another formulation of the problem in order to obtain efficient numerical results.

3 Relaxation Technique of the Robust and Arbitrary Pole Placement problem using SOF: TRAPP-SOF

In section 2, we showed that robust and arbitrary pole placement with SOF can be expressed by a problem (12). In general, the resolution of the robust pole placement problem with output feedback control is very complicated to solve. To reduce the complexity of the problem, we propose a relaxation technique of the cost function in this section, the idea of this technique is inspired by the famous work [8].
Let X is a non singular matrix depicted by the vectors satisfying

$$||x_i||^2 = \sum_{j=1}^{n} \left|x_{i,j}\right|^2 = 1. \tag{13}$$

In this section, it is assumed that the vectors x_i constituent the columns of the matrix X have unit length. For this purpose, the following unit set $(\mathbb{R}^n, ||.||)$ is defined:

$$E := \{x \in \mathbb{R}^n : ||x||^2 = 1\} \tag{14}$$

Where each x_i is contained in E.

Theorem 3 *The solution of the Robust and Arbitrary Pole Placement problem with static output feedback control of linear systems is given by the following optimization problem:*

$$(TRAPP - SOF) \quad \begin{cases} \min_{x_i \in E} trace(X^{-1}X^{-T}) \\ subject\ to \\ \quad (\mathbb{I} - BB^{+})(AX - X\Sigma) = 0 \\ \quad B^{+}(A - X\Sigma X^{-1})(C^{+}C - \mathbb{I}) = 0 \end{cases}$$

Proof. *The proof is straightforward from the result of Theorem (2) and the following lemma [4]*

Lemma 4 *[4]*
Let X is a non singular matrix depicted by the vectors satisfying (13). Then

$$\min_{x_i \in E} Cond_F(X) = \min_{x_i \in E} trace(Q^{-1})$$

Where $Q = \frac{1}{n} X^T X$

■

4 Numerical implementation

Beyond the past and present results of robust pole placement with output feedback, it seems that the theory of robust control needs to take even better advantage of the theoretical and numerical developments in mathematical programming that are offered by several software packages.

The problem (RAPP-SOF) is a non-convex and nonlinear optimization problem with non-linear equality constraints, this problem is hard to solve numerically. So, it can be solved by using : iterative Sequential Quadratic Programming (SQP) [3] such as Sequential Quadratic Programming legacy (SQP-legacy) [9], Interior-point (IP) [2] or Active-set (AS) [5]. However, one proposes the interior-point method [7] witch is the best approach of the resolution for this problem for the following reasons:
• It offers a great simplicity in the treatment of inequality constraints by the Logarithmic Barrier Function.

• It converges quickly to the optimal solution.
• It does not require a choice of starting point.
In the following section, for the illustrative examples, we directly employ the interior-point algorithm of fmincon function in Matlab 2021a. fmincon function is a built-in program in Matlab for to solve the nonlinear problems (RAPP-SOF) and (TRAPP-SOF). This function can be used to search and find all possible local minima that satisfy the given objective. The iteration process starts with an initial guess by the algorithm, and stops when all setup criteria are met. If the first-order optimisation is fulfilled by the last iteration, the result is considered as a local minimum that satisfies the constraints.
Several papers in the literature propose theoretical solutions to the robust pole placement problem with SOF without numerical implementation, or sometimes only with one or two illustrative examples. In this work, the proposed theoretical approaches (RAPP-SOF) and (TRAPP-SOF) are simulated on the basis of specific irregular examples collected from different works, We are going to keep the same poles that were chosen by the authors to be placed.
The methods are implemented using Matlab 2021a and all computations were carried out on a Laptop Core(TM) $I7/7^{th}$ GEN with 2.80 GHz and 16 GB RAM.

Example 1

Consider the system studied by [6]:

$$A = \begin{bmatrix} -11.4 & -3.5 & 0 \\ 4 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 & 1 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 1.425 \\ 1 & -1 & 0 \end{bmatrix}.$$

The open-loop system has eigenvalues: $\{0, -1.4, -10\}$.
Choose $\{-1, \ -2, \ -3\}$ the poles to be assigned. In this example, other arbitrary poles can be placed robustly with success.

Example 2

Consider the second open-loop system studied by [6]

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 19.62 & 0 & -8.86 \\ 0 & 0 & -100 \end{bmatrix}, B = \begin{bmatrix} 0 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}.$$

The open-loop system has eigenvalues:$\{4.42, -4.42, -100\}$. The spectrum to be placed is: $\sigma = \{-3, \ -3, \ -4\}$. Furthermore, we can robustly place any pole as separate, multiple or overlapping with the open-loop eigenvalues.

Example 3

Consider the open-loop system studied by [13]

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

The open-loop system has eigenvalues: $\{1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\}$.
Take $\{1, \pm j\}$ the same spectrum that was chosen in [13] to be robustly assigned. In general, solving the problem (RAPP-SOF) or (TRAPP-SOF) is not feasible for other arbitrary poles.

Example 4

Consider the open-loop system studied in [13]

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

The open-loop system has eigenvalues:$\{-4, -3, 1, 2\}$.
Let $\{-1, -2, -3, -5\}$ the poles to be assigned. Arbitrary poles can be successfully placed with an optimal condition number from the resolution of the proposed approaches.
The numerical results of the condition number by solving the optimization problem (RAPP-SOF) for the selected examples above is as follow:

Table 1. Results of numerical processing.

Examples	RAPP-SOF	
	$Cond_F(X)$	$Cond_2(X)$
1	50.117	48.924
2	6960	6957
3	3.674	2
4	73.664	59.378

For the numerical resolution of the optimization problem (TRAPP-SOF) the simulation is done on the same examples treated above. The results of the numerical treatment of two proposed

Table 2. Numerical results of the relaxation technique.

Examples	TRAPP-SOF	
	$Cond_F(X)$	$Cond_2(X)$
1	42.718	41.243
2	676.390	675.291
3	3.674	2
4	40.073	29.558

methods presented in Table 1 and Table 2, by applying the interior-point algorithm of fmincon function, show that the best performances were provided by the (TRAPP-SOF) method.

5 Conclusion

In this paper, an arbitrary pole placement problem has been treated using static output feedback control, where the desired closed-loop poles can be placed robustly and accurately. In order to compute a well-conditioned solution, the Robust and Arbitrary Pole placement (RAPP) problem has been adressed to minimise the condition number in the frobenuis norm. Two proposed techniques were used to formulate the RAPP as constrained non-linear and non-convex optimization problems to be solved by the interior point algorithm of the *fmincon* function in Matlab. A numerical and comparative study of the proposed methods was performed on irregular litterature examples.

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