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Adaptive threshold scheme for fault detection for positive switched systems with parametric uncertainty

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Abstract. This paper addresses the challenge of designing a fault detection observer for actuators and sensors in a particular class of positive switched systems, using the average dwell time method. The proposed approach involves the concurrent generation of residuals and a dynamic threshold, enhancing the precision and efficiency of fault detection. Additionally, our work introduces an innovative method for fault detection with parametric uncertainties, outlining sufficient conditions for designing the observer using Linear Programs (LP). The effectiveness of the proposed approach is validated through a numerical simulation.

1 Introduction

Over the past few decades, fault detection (FD) has gained significant attention, particularly in the study of positive switched systems [1], [2], [3]. These systems, which operate by switching between different modes while ensuring that their state variables remain non-negative, are widely used in many practical applications, such as biological processes and networked systems. Detecting faults in such systems is a challenging task due to the complex dynamics introduced by arbitrary switching sequences. Furthermore, when considering parametric uncertainty in fault detection, it is essential to use a time-varying threshold in order to build a reliable FD strategy that can dynamically adapt to changing system conditions, including the impacts of uncertain parameters. This variability of the threshold allows for the maintenance of robustness and reliability in the fault detection process, even when system parameters are uncertain or variable. However, if the diagnostic threshold is designed without considering the effects of these variabilities, it may not accurately reflect the system's behavior and could lead to false alarms.

It should be noted that the average dwell time (ADT) technique has become a popular method for solving several control problems including positive switched systems [4, 5]. The average dwell-time technique means that the number of switches in the finite interval is bounded and the average time between successive switching is greater than or equal to a positive constant, then, the switched system is exponentially stable [6, 7]. Actually, there are few results reported on deploying average dwell-time techniques to fault detection of positive switched systems [8], but in the literature, various works have been treated the problem of FD detection for general switched systems while satisfying the ADT and DT

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constraints. For example, [9] has been established delay-dependent sufficient condition for the solvability of fault detection for a class of discrete time switched systems using Lyapunov functional method. More recently, fault detection problem for switched systems under dwell time constraints has been proposed in [3] using the parity space-based approach. Further, [10] has been investigated the issues of stability analysis and state feedback control for a class of discrete time switched systems via the smooth approximation technique.

Motivated by the previous discussion, we direct our attention to time-varying threshold-based fault detection for uncertain positive switched systems considering the average dwell time approach. The fault detection strategy is based on time-varying thresholds, which can enhance the robustness of fault detection methods by allowing systems to adapt to various uncertainties. Thus, our work proposes a mixed $L-/L_1$ fault detection strategy to address faults, parametric uncertainties, and unknown inputs for positive switched systems with state delay. By employing a multiple Lyapunov function, we derive sufficient conditions for the existence of such observer through linear programming techniques.

2 PROBLEM FORMULATION

The following positive switched system with state delay is given as

$$x_{k+1} = \sum_{i=1}^{N} \xi_i(k)[\mathcal{A}_i x_k + \mathcal{A}_\tau x_{k-\tau} + \mathcal{E}_i d_k + F_i f_k] \tag{1}$$

$$y_k = \sum_{i=1}^{N} \xi_i(k)[C_i x_k + \mathcal{N}_i d_k + \mathcal{M}_i f_k] \tag{2}$$

$$x_k = \underline{\phi}_k \geq 0, \tag{3}$$

where $x_k \in \mathbb{R}^n$ denotes the state, $y_k \in \mathbb{R}^p$ represents the output, $d_k \in \mathbb{R}^g$, $f_k \in \mathbb{R}^h$ and τ correspond to the disturbance input and fault vector, respectively. The initial conditions are represented by $\underline{\phi}_k \geq 0$. σ_k denotes the switching rule. $\xi(k) = [\xi_1(k), \ldots, \xi_N(k)]^T$ is the indicator function, with $\xi_i(k) = 1$ if the switching system is in mode i and 0 elsewhere.

Assumption 2.1 *In this work, the following assumptions are required:*

- *The switching rule is not known a priori but σ_k is available at each time k, ($k \in \mathbb{Z}_+$).*
- *Each time only one subsystem is active*
- *The pair $(\mathcal{A}_i, C_i)$ is observable.*

Definition 2.1 *If for any switching sequences σ_k, for any non-negative initial conditions $\underline{\phi}_k \geq 0$, non-negative unknown inputs $d_k \geq 0$, any fault $f_k \geq 0$, the corresponding trajectories $x_k \geq 0$ and y_k remain non-negative for all $k \geq 0$.*

[6] For any $k_2 > k_1 > 0$ and any switching signal σ_k, let $N_\sigma(\tau, k)$ represent the number of switching of σ_k within the interval (k_1, k_2). If

$$N_\sigma(\tau, k) \leq N_0 + \frac{k_2 - k_1}{\tau_a} \tag{4}$$

holds for a given $N_0 \geq 0$ and τ_a, where, τ_a is noted the average dwell time and N_0 represent the chattering bound.

Lemma 2.1 *If and only if $\mathcal{A}_{\tau i} \geq 0$, $\mathcal{A}_i \geq 0$, $C_i \geq 0$, $\mathcal{E}_i \geq 0$, $\mathcal{N}_i \geq 0$, $\mathcal{M}_i \geq 0$ and $\mathcal{F}_i \geq 0$, $\forall i \in \underline{\mathcal{I}}$, then, system (1) is positive.*

Our focus is now on developing a Luenberger observer for fault detection. The switched positive observers being analyzed are described by:

$$\begin{aligned}\hat{x}_{k+1} &= \sum_{i=1}^{N} \xi_i(k)[\mathcal{A}_i \hat{x}_k + A_{\tau i}\hat{x}_{k-\tau} + \mathcal{L}_i C_i e_k] \\ \hat{y}_k &= \sum_{i=1}^{N} \xi_i(k) C_i \hat{x}_k \end{aligned} \tag{5}$$

The residual for the observer is given as:

$$r_k = y_k - \hat{y}_k \tag{6}$$

Defining the observer error by

$$\begin{aligned} e_k &= x_k - \hat{x}_k \\ e_{k+1} &= \sum_{i=1}^{N} \xi_i(k)[(\mathcal{A}_i - \mathcal{L}_i C_i)e_k + A_{\tau i} e_{k-\tau}) + (E_i - \mathcal{L}_i N_i)d_k + (F_i - \mathcal{L}_i \mathcal{M}_i) f_k] \end{aligned} \tag{7}$$

Let $\tilde{x}_k = \begin{bmatrix} x_k \\ e_k \end{bmatrix}$ represent the augmented state vector. The corresponding dynamical system can be expressed as

$$\begin{aligned}\tilde{x}_{k+1} &= \sum_{i=1}^{N} \xi_i(k)[\tilde{\mathcal{A}}_i \tilde{x}_k + \tilde{\mathcal{A}}_{\tau i}\tilde{x}_{k-\tau} + \tilde{\mathcal{E}}_i d_k + \tilde{\mathcal{F}}_i f_k], \\ \tilde{r}_k &= \sum_{i=1}^{N} \xi_i(k)[\tilde{C}_i \tilde{x}_k + \tilde{\mathcal{N}}_i d_k + \tilde{\mathcal{M}}_i f_k], \end{aligned} \tag{8}$$

where

$$\begin{aligned}\tilde{\mathcal{A}}_i &= \begin{bmatrix} \mathcal{A}_i & 0 \\ 0 & \mathcal{A}_i - \mathcal{L}_i C_i \end{bmatrix}, \tilde{\mathcal{A}}_{\tau i} = \begin{bmatrix} \mathcal{A}_{\tau i} & 0 \\ 0 & \mathcal{A}_{\tau i} \end{bmatrix}, \tilde{\mathcal{E}}_i = \begin{bmatrix} \mathcal{E}_i \\ \mathcal{E}_i - \mathcal{L}_i \mathcal{N}_i \end{bmatrix}, \\ \tilde{\mathcal{F}}_i &= \begin{bmatrix} \mathcal{F}_i \\ \mathcal{F}_i - \mathcal{L}_i \mathcal{M}_i \end{bmatrix}, \tilde{C}_i = \begin{bmatrix} 0 & C_i \end{bmatrix}, \tilde{\mathcal{N}}_i = \mathcal{N}_i, \tilde{\mathcal{M}}_i = \mathcal{M}_i \end{aligned}$$

3 PRELIMINARY RESULTS

For fault detection purposes, estimating the fault f_k is not required. Instead, the following residual criterion can be utilized:

$$J_k = \left(\sum_{s=k_0}^{k_0+k} r_s^T r_s \right)^{1/2}, \tag{9}$$

where k_0 and k represents the initial evaluation time instant and the evaluation time steps, respectively. After selecting the evaluation function, the threshold can be determined. Faults can be detected based on:

$$\begin{aligned} J_k &> J_{th}(k) \Rightarrow \text{faults} \Rightarrow \text{alarm} \\ J_k &\leq J_{th}(k) \Rightarrow \text{no faults} \end{aligned}$$

time varying threshold is given as:

$$J_{th}(k) = \sup_{d_k \in L_1, f_k = 0,} J_k^0 \tag{10}$$

where J_k^0 represents a criterion obtained with a residual r_0 calculated without faults.

Definition 3.1 *[8] System (8) is said to achieve a performance index $\underline{\gamma}$, if $\tilde{x}_k = 0$, $k = [-\tau, 0]$ and for given positive scalars $\underline{\gamma}$ and $0 < \tilde{\zeta} < 1$, then*

$$\sup_{d_k \neq 0, d_k \in L_1[0,\infty)} \frac{\sum_{k=k_o}^{\infty} (1-\tilde{\zeta})^{(k-k_o)} \| r_k \|_1}{\sum_{k=k_o}^{\infty} \| d_k \|_1} < \underline{\gamma} \tag{11}$$

Definition 3.2 *system (8) is said to have a performance index β, if $\tilde{x}_k = 0$, $k = [-\tau, 0]$ and for given positive scalars $\underline{\beta}$ and $0 < \tilde{\zeta} < 1$, such that [8]:*

$$\inf_{f_k \neq 0, f_k \in L_1[0,\infty)} \frac{\sum_{k=k_o}^{\infty} \| r_k \|_1}{\sum_{k=k_o}^{\infty} (1-\zeta)^{(k-k_o)} \| f_k \|_1} > \underline{\beta} \tag{12}$$

Lemma 3.1 *[11] If and only if $\mathcal{L}_i C_i \geq 0$, $(\mathcal{A}_i - \mathcal{L}_i C_i) \geq 0$, and $(\mathcal{A}_i - \mathcal{L}_i C_i)$ is a schur matrix. Then, the observer in equation (5) is a convergent positive observer for system (1).*

4 Main results

First of all, we present the stability condition of the augmented system (8).

Lemma 4.1 *For given positive constants $0 < \tilde{\zeta} < 1$ and $\tilde{\iota} \geqslant 1$, system (8) is asymptotically stable, if there exist positive vectors $\underline{\varrho}_i$ and $\underline{\varphi}_i$ such that*

$$((\tilde{\mathcal{A}}_i^T - \mathbb{I}) + \tilde{\zeta}\mathbb{I})\underline{\varrho}_i + (1-\tilde{\zeta})\varphi_i < 0 \tag{13}$$

$$\tilde{\mathcal{A}}_{\tau i}^T \underline{\varrho}_i - (1-\tilde{\zeta})^{\tau+1} \underline{\varphi}_i < 0 \tag{14}$$

$$\underline{\varrho}_i \leqslant \tilde{\iota}\underline{\varrho}_j, \quad \underline{\varphi}_i \leqslant \tilde{\iota}\underline{\varphi}_j \tag{15}$$

while the ADT satisfying

$$\tau_a \geq \tau_a^* = -\frac{ln\tilde{\iota}}{ln(1-\tilde{\zeta})} \tag{16}$$

Proof 1 *Choosing the following multiple co-positive Lyapunov-Krasovskii function, one can obtain*

$$\underline{V}_i(\tilde{x}_k) = \tilde{x}_k^T \underline{\varrho}_i + \sum_{s=k-\tau}^{k-1} (1-\tilde{\zeta})^{k-s} \tilde{x}_s^T \underline{\varphi_i} \tag{17}$$

$$\begin{aligned} \Delta \underline{V}_i(\tilde{x}_k) + \tilde{\zeta}\underline{V}_i(\tilde{x}_k) &= \tilde{x}_{k+1}^T \underline{\varrho}_i - \tilde{x}_k^T \underline{\varrho}_i + \sum_{s=k+1-\tau}^{k} (1-\tilde{\zeta})^{k+1-s} \tilde{x}_s^T \underline{\varphi}_i - \sum_{s=k-\tau}^{k-1} (1-\tilde{\zeta})^{k-s} \tilde{x}_s^T \underline{\varphi}_i \\ &+ \tilde{\zeta}\tilde{x}_k^T \underline{\varrho}_i + \tilde{\zeta} \sum_{s=k-\tau}^{k-1} (1-\tilde{\zeta})^{k-s} \tilde{x}_s^T \underline{\varphi}_i \leq \sum_{i=1}^{N} \xi_i(k) \left\{ \tilde{x}_k^T [(\tilde{\mathcal{A}}_i^T - \mathbb{I}) + \tilde{\zeta}\mathbb{I})\underline{\varrho}_i \right. \\ &+ \left. (1-\tilde{\zeta})\underline{\varphi}_i] + \tilde{x}_{k-\tau}^T [\tilde{\mathcal{A}}_{\tau i}^T \underline{\varrho}_i - (1-\tilde{\zeta})^{\tau+1})\underline{\varphi}_i] \right\} \end{aligned} \tag{18}$$

This results from (13)-(15) and (18) that

$$\Delta\underline{V}_i(\tilde{x}_k) + \tilde{\zeta}\underline{V}_i(\tilde{x}_k) \leq 0 \Rightarrow \underline{V}_i(\tilde{x}_{k+1}) \leq (1-\tilde{\zeta})\underline{V}_i(\tilde{x}_k) \tag{19}$$

Hence, along the trajectory of System (8), we hold

$$\underline{V}_i(\tilde{x}_k) \leqslant (1-\tilde{\underline{\zeta}})^{k-k_o^i}\underline{V}_i(\tilde{x}_{k_o^i}), \quad k > k_o^i \tag{20}$$

where k_o^i represents the initial of the i^{th} activated subsystem.
Let $k_o < k_1 < k_2 < ... < k_g < k_{g+1}... < k$ represent the switching instants of σ_k, when $k \in [k_g, k_{g+1})$, one can get

$$\underline{V}_i(\tilde{x}(k_{g+1})) \leqslant \underline{V}_i(\tilde{x}(k_g))(1-\underline{\zeta})^{(k_{g+1}-k_g)} \tag{21}$$

Then, over the interval $[k_o, k)$, and based on (15) *and* (21)*, we derive*

$$\begin{aligned}\underline{V}_i(\tilde{x}_k) &\leqslant (1-\tilde{\zeta})^{(k-k_g)}\underline{V}_i(\tilde{x}(k_g)) \\ &\leqslant \tilde{\iota}(1-\tilde{\zeta})^{k-k_{\bar{g}}}\underline{V}_{i-1}(\tilde{x}(k_{\bar{g}})) \\ &\leqslant ... \leqslant \tilde{\iota}^{N_{\sigma(k_o,k)}}(1-\underline{\tilde{\zeta}})^{k-k_o}\underline{V}_o(\tilde{x}(k_o))\end{aligned} \tag{22}$$

the switching instant just before k_g is denoted as $k_{\bar{g}}$.
According to $N_\sigma(k_o,k) \leqslant N_o + \frac{k-k_o}{\tau_a}$. Then, (22) *becomes:*

$$\begin{aligned}\underline{V}_i(\tilde{x}_k) &\leqslant \tilde{\iota}^{N_o}\tilde{\iota}^{(k-k_o)\tau_a}(1-\tilde{\zeta})^{k-k_o}\underline{V}_o(\tilde{x}(k_o)) \\ &\leqslant \left[\tilde{\iota}^{\frac{1}{\tau_a}}(1-\tilde{\zeta})\right]^{k-k_o}\underline{V}_o(\tilde{x}(k_o))\end{aligned} \tag{23}$$

Thus, $\underline{V}_i(\tilde{x}_k) \to 0$ when $k \to \infty$ and if ADT guaranteeing (16)*.*

Next, the L_1 index for disturbance attenuation is provided.

Lemma 4.2 *System (8) is asymptotically stable and satisfying* (16)*, if there exist vectors $\underline{\varrho}_i$ and $\underline{\varphi}_i$ and for given positive scalars $\tilde{\iota} \geqslant 1, \underline{\gamma}$ and $0 < \tilde{\zeta} < 1$, such that*

$$((\tilde{\mathcal{A}}_i^T - \mathbb{I}) + \tilde{\zeta}\mathbb{I})\underline{\varrho}_i + (1-\tilde{\zeta})\underline{\varphi}_i + \tilde{C}_i^T\boldsymbol{e} < 0 \tag{24}$$

$$\tilde{\mathcal{A}}_{\tau i}^T\underline{\varrho}_i - (1-\tilde{\zeta})^{\tau+1}\underline{\varphi}_i < 0 \tag{25}$$

$$\tilde{\mathcal{E}}_i^T\underline{\varrho}_i - \underline{\gamma}\boldsymbol{e} + \tilde{\mathcal{N}}_i^T\boldsymbol{e} < 0 \tag{26}$$

$$(\mathcal{A}_i - \mathcal{L}_iC_i) \geq 0 \tag{27}$$

$$(\mathcal{E}_i - \mathcal{L}_iN_i) \geq 0 \tag{28}$$

$$\mathcal{L}_iC_i \geq 0 \tag{29}$$

Proof 2 *According to Lemma 4.1, the asymptotic stability of System (8) with $d_k = 0$ is guaranteed if conditions (24) and (25) hold. To demonstrate the performance* (11)*, we choose (17). From (15), we derive the following results:*

$$\underline{V}_{\sigma(k_g)} \leqslant \tilde{\iota}\underline{V}_{\sigma(k_{\bar{g}})}(k_{\bar{g}}), \quad g = 1, 2, ... \tag{30}$$

For any $k \in [k_g, k_{g+1})$, we have

$$\underline{V}_{\sigma(k)}(k) \leqslant (1-\tilde{\zeta})^{k-k_g}\underline{V}_{\sigma(k_g)} - \sum_{s=k_g}^{k-1}(1-\tilde{\zeta})^{k-s-1}\tilde{\Pi}(s) \tag{31}$$

where $\tilde{\Pi}(s) = \| r_s \|_1 - \underline{\gamma} \| d_s \|_1$. *Combining* (30) *and* (31) *leads to*

$$\begin{aligned}
\underline{V}_{\sigma(k)}(k) &\leqslant \tilde{\iota}(1-\tilde{\zeta})^{(k-k_g)}\underline{V}_{\sigma(k_{\bar{g}})}(k_{\bar{g}}) - \sum_{s=k_g}^{k-1}(1-\tilde{\zeta})^{(k-s-1)}\tilde{\Pi}(s) \\
&\leqslant \tilde{\iota}(1-\tilde{\zeta})^{(k-k_{g-1})}\underline{V}_{\sigma(k_{g-1})}(k_{g-1}) \\
&- \tilde{\iota}\sum_{s=k_{g-1}}^{k_g}(1-\tilde{\zeta})^{k-s-1}\tilde{\Pi}(s) - \sum_{s=k_g}^{k-1}(1-\tilde{\zeta})^{k-s-1}\tilde{\Pi}(s) \\
\leqslant \ldots &\leqslant \tilde{\iota}^{N_\sigma(k_o,k)}(1-\tilde{\zeta})^{(k-k_o)}V_{\sigma(k_o)}(k_o) \\
&- \tilde{\iota}^{N_\sigma(k_o,k)}\sum_{s=k_o}^{k_1-1}(1-\tilde{\zeta})^{(k-s-1)}\tilde{\Pi}(s) - \ldots - \sum_{s=k_g}^{k-1}(1-\tilde{\zeta})^{(k-s-1)}\tilde{\Pi}(s) \\
&= \tilde{\iota}^{N_\sigma(k_o,k)}(1-\tilde{\zeta})^{(k_o-k)}V_{\sigma(k_o)}(k_o) - \sum_{s=k_o}^{k-1}\tilde{\iota}^{N_{\sigma(s,k)}}(1-\tilde{\zeta})^{(k-s-1)}\tilde{\Pi}(s)
\end{aligned} \tag{32}$$

then, (32) *indicates*

$$\sum_{s=k_o}^{k-1}\tilde{\iota}^{N_{\sigma(s,k)}}(1-\tilde{\zeta})^{(k-s-1)} \| r_s \|_1 \leq \underline{\gamma}\sum_{s=k_o}^{k-1}\tilde{\iota}^{N_{\sigma(s,k)}}(1-\tilde{\zeta})^{(k-s-1)} \| d_s \|_1 \tag{33}$$

Multiplying both sides of (33) *by* $\tilde{\iota}^{-N_{\sigma(k_o,k)}}$, *we get*

$$\sum_{s=k_o}^{k-1}\tilde{\iota}^{N_{\sigma(k_o,s)}}(1-\tilde{\zeta})^{(k-s-1)} \| r_s \|_1 \leq \underline{\gamma}\sum_{s=k_o}^{k-1}\tilde{\iota}^{N_{\sigma(k_o,s)}}(1-\tilde{\zeta})^{(k-s-1)} \| d_s \|_1 \tag{34}$$

Notice that

$$N_\sigma(k_o, s) \leq (s-k_o)/\tau_a$$

and

$$\tau_a \geq \tau_a^* = -\frac{ln\tilde{\iota}}{ln(1-\tilde{\zeta})}$$

we have

$$\tilde{\iota}^{-N_\sigma(k_o,s)} \leq (1-\tilde{\zeta})^{(s-k_o)}$$

Hence, (34) *implies*

$$\sum_{s=k_o}^{k-1}(1-\tilde{\zeta})^{(s-k_o)}1-\tilde{\zeta})^{(k-s-1)} \| r_s \|_1 \leq \underline{\gamma}\sum_{s=k_o}^{k-1}(1-\tilde{\zeta})^{(k-s-1)} \| d_s \|_1 \tag{35}$$

Summing (35) *from* $k = k_o$ *to* ∞ *leads to Inequality* (11). □

In what follows, the $L\_$ index for disturbance attenuation is provided.

Lemma 4.3 *System (8) is positive and guarantees (12), (39)-(40) and (16), if there exist vectors $\bar{\varrho}_i$ and $\bar{\varphi}_i$ and for given positive scalars $\tilde{\iota} \geq 1$, $\underline{\beta}$ and $0 < \tilde{\zeta} < 1$, such that*

$$((\tilde{\mathcal{A}}_i^T - \mathbb{I}) + \tilde{\zeta}\mathbb{I})\bar{\varrho}_i + (1-\tilde{\zeta})\bar{\varphi}_i - \tilde{C}_i^T \boldsymbol{e} < 0 \tag{36}$$

$$\tilde{\mathcal{A}}_{\tau i}^T \bar{\varrho}_i - (1-\tilde{\zeta})^{\tau+1}\bar{\varphi}_i < 0 \tag{37}$$

$$\tilde{\mathcal{F}}_i^T \bar{\varrho}_i - \tilde{\mathcal{M}}_i^T \boldsymbol{e} + \underline{\beta}\boldsymbol{e} < 0 \tag{38}$$

$$(\mathcal{A}_i - \mathcal{L}_i C_i) \geq 0 \tag{39}$$

$$\mathcal{L}_i C_i \geq 0 \tag{40}$$

$$(F_i - \mathcal{L}_i \mathcal{M}_i) \geq 0 \tag{41}$$

$$\bar{\varrho}_i \leqslant \tilde{\iota}\bar{\varrho}_j, \quad \bar{\varphi}_i \leqslant \tilde{\iota}\bar{\varphi}_j \tag{42}$$

Proof 3 *Consider (17) and denoting $k_o < k_1 < k_2 < ... < k_g < k_{g+1}... < k$ the switching instants on the interval $[k_o, k)$. Similar to the proof line of Lemma 4.1 , one get:*

$$\Delta \underline{V}_k/_{d_k=0} + \tilde{\zeta}\underline{V}_k + \underline{\beta}\|f_k\|_1 - \|r_k\|_1 \leq \sum_{i=1}^{N} \xi_i(k) \Big\{ \tilde{x}_k^T (\tilde{\mathcal{A}}_i^T \bar{\varrho}_i - \bar{\varrho}_i + \tilde{\zeta}\bar{\varrho}_i + (1-\tilde{\zeta})\bar{\varphi}_i - \tilde{C}_i^T \boldsymbol{e}) + \tilde{x}_{k-\tau}^T (\tilde{\mathcal{A}}_{\tau i}^T \bar{\varrho}_i - (1-\tilde{\zeta})^{\tau+1}\bar{\varphi}_i) + f_k^T (\tilde{\mathcal{F}}_i^T \bar{\varrho}_i - \tilde{\mathcal{M}}_i^T \boldsymbol{e} + \underline{\beta}\boldsymbol{e}) \Big\}$$

It follows from (36)-(38) that

$$\Delta \underline{V}_k/_{d_k=0} + \tilde{\zeta}\underline{V}_k - \|r_k\|_1 + \underline{\beta}\|f_k\|_1 < 0 \tag{43}$$

yields to

$$\underline{V}(k) \leqslant (1-\tilde{\zeta})^{(k-k_g)} V_{\sigma(k_g)}(k_g) - \sum_{s=k_g}^{k-1} (1-\tilde{\zeta})^{(k-s-1)} \tilde{\Pi}(s) \tag{44}$$

where $\tilde{\Pi}(s) = \underline{\beta} \parallel f_k \parallel_1 - \parallel r_k \parallel_1$. By referencing the proof outline of Lemma 4.2, we derive

$$\sum_{k=k_o}^{\infty} (1-\tilde{\zeta})^{(k-k_o)} \underline{\beta} \parallel f_k \parallel_1 \leq \sum_{k=k_o}^{\infty} \parallel r_k \parallel_1 \tag{45}$$

Then, the performance (12) is satisfied. □

Now, we design a positive observer for fault detection based on $L\_/L_1$ norms.

Lemma 4.4 *For a given positive scalar $\underline{\beta}$, if there exist vectors $\bar{\varrho}_i = \underline{\varrho}_i \in \mathbb{R}_+^{2n}$ and $\bar{\varphi}_i = \underline{\varphi}_i \in \mathbb{R}_+^{2n}$ such that (13)-(14) and (38) hold, thus there exists a stable L_fault detection observer in the form of (5) with the ADT* (16)*, where $\tilde{\iota} \geq 1$ is satisfying (15).*

Proof 4 *Obvious.* □

Lemma 4.5 *There exists a stable $L\_/L_1$ fault detection positive observer satisfying* (16)*, where $\tilde{\iota} \geq 1$ satisfies (15), if if there exist vectors $\bar{\varrho}_i = \underline{\varrho}_i \in \mathbb{R}_+^{2n}$ and $\bar{\varphi}_i = \underline{\varphi}_i \in \mathbb{R}_+^{2n}$ and for given positive scalars $\underline{\beta}$, $\underline{\gamma}$ and $0 < \tilde{\zeta} < 1$ such that (24)-(26) and (38) hold.*

Proof 5 *Noting that (36)-(37) can be directly obtained from (24)-(25) for $\bar{\bar{\varrho}}_i = \underline{\varrho}_i \in \mathbb{R}_+^{2n}$ and $\bar{\varphi}_i = \underline{\varphi}_i \in \mathbb{R}_+^{2n}$. Then, the above lemma holds.* □

Theorem 4.1 *For given $\underline{\beta} > 0$, $\underline{\gamma} > 0$, $0 < \tilde{\zeta} < 1$ and $\tilde{\iota} \geq 1$, if there exist vectors $z_1^i, \cdots, z_n^i \in \mathbb{R}^h$, $\underline{\vartheta}_i > 0$, $\underline{\nu}_i > 0$, $\underline{\kappa}_i > 0$ and $\underline{\eta}_i > 0$, such that the following Linear Programming is feasible:*

$$
\begin{aligned}
(A_i^T - \mathbb{I})\underline{\nu}_i + \tilde{\zeta}\underline{\nu}_i - C_i^T \sum_{l=1}^{n} z_l^i + (1-\tilde{\zeta})\underline{\eta}_i + C_i^T \boldsymbol{e} &< 0 && (46)\\
F_i^T(\underline{\kappa}_i + \underline{\nu}_i) - M_i^T \sum_{l=1}^{n} z_l^i - M_i^T \boldsymbol{e} + \underline{\beta}\boldsymbol{e} &< 0 && (47)\\
E_i^T(\underline{\kappa}_i + \underline{\nu}_i) - N_i^T \sum_{l=1}^{n} z_l^i - N_i^T \boldsymbol{e} + \underline{\gamma}\boldsymbol{e} &< 0 && (48)\\
(A_i^T - \mathbb{I})\underline{\kappa}_i + \tilde{\zeta}\underline{\kappa}_i + (1-\tilde{\zeta})\underline{\vartheta}_i &< 0 && (49)\\
A_{\tau i}^T \underline{\kappa}_i - (1-\tilde{\zeta})^{\tau+1}\underline{\vartheta}_i &< 0 && (50)\\
A_{\tau i}^T \underline{\nu}_i - (1-\tilde{\zeta})^{\tau+1}\underline{\eta}_i &< 0 && (51)\\
a_{ql}^i \underline{\nu}_q - c_l^{iT} z_q^i &\geq 0 && (52)\\
e_{js}^i \underline{\nu}_q - n_s^{iT} z_q^i &\geq 0 && (53)\\
f_{qk}^i \underline{\nu}_q - m_k^{iT} z_q^i &\geq 0 && (54)\\
c_l^{iT} z_q^i &\geq 0 && (55)\\
\underline{\eta}_i &\leq \tilde{\iota}\underline{\eta}_j && (56)\\
\underline{\kappa}_i &\leq \tilde{\iota}\underline{\kappa}_j && (57)\\
\underline{\nu}_i &\leq \tilde{\iota}\underline{\nu}_j && (58)\\
z_l^i &> 0 && (59)
\end{aligned}
$$

such as

$$
\begin{aligned}
&for \quad l, q = 1, \ldots, n\\
&for \quad s = 1, \ldots, g\\
&for \quad k = 1, \ldots, h
\end{aligned}
$$

where a_{lj}^i, e_{js}^i and f_{jk}^i represent respectively, the elements of matrices $\mathcal{E}_i$, $\mathcal{A}_i$ and $\mathcal{F}_i$. c_l^i, n_s^i and m_s^i are the column vectors of matrices $N_i = [n_1^i, \ldots, n_g^i]$, $C_i = [c_1^i, \ldots, c_n^i]$ and $\mathcal{M}_i = [m_1^i, \ldots, m_h^i]$, respectively. Then, a mixed $L\_/L_1$ fault detection positive observer is formulated while satisfying (16). *where*

$$
\mathcal{L}_i = [\frac{z_1^i}{\underline{\nu}_1^i}, \frac{z_2^i}{\underline{\nu}_2^i}, \ldots, \frac{z_n^i}{\underline{\nu}_n^i}]^T. \qquad (60)
$$

Proof 6 *Denote*

$$
\underline{\varrho}_i = \begin{bmatrix} \underline{\kappa}_i^T \\ \underline{\nu}_i^T \end{bmatrix}, \underline{\varphi}_i = \begin{bmatrix} \underline{\vartheta}_i^T \\ \underline{\eta}_i^T \end{bmatrix},
$$

Substituting into (24)-(26) and (38), we get

$$(\mathcal{A}_i^T - \mathbb{I})\underline{\nu}_i + \tilde{\zeta}\underline{\nu}_i - C_i^T \sum_{l=1}^{n} z_l^i + C_i^T \boldsymbol{e} + (1-\tilde{\zeta})\underline{\eta}_i < 0 \tag{61}$$

$$F_i^T(\underline{\kappa}_i + \underline{\nu}_i) - \mathcal{M}_i^T \sum_{l=1}^{n} z_l^i - M_i^T \boldsymbol{e} + \underline{\beta}\boldsymbol{e} < 0 \tag{62}$$

$$\mathcal{E}_i^T(\underline{\kappa}_i + \underline{\nu}_i) - \mathcal{N}_i^T \sum_{l=1}^{n} z_l^i + \underline{\gamma}\boldsymbol{e} - \mathcal{N}_i^T \boldsymbol{e} < 0 \tag{63}$$

$$(\mathcal{A}_i^T - \mathbb{I})\underline{\kappa}_i + \tilde{\zeta}\underline{\kappa}_i + (1-\tilde{\zeta})\underline{\vartheta}_i < 0 \tag{64}$$

$$\mathcal{A}_{\tau i}^T\underline{\kappa}_i - (1-\tilde{\zeta})^{\tau+1}\underline{\vartheta}_i < 0 \tag{65}$$

$$\mathcal{A}_{\tau i}^T\underline{\nu}_i - (1-\tilde{\zeta})^{\tau+1}\underline{\eta}_i < 0 \tag{66}$$

Equation (60) can be expressed as:

$$\mathcal{L}_i^T \underline{\nu}_i = \sum_{l=1}^{n} z_l^i \tag{67}$$

Consequently, (46)-(51) can be reformulated as (61)-(66). Now let us prove that (39) and (52) are equivalent. Note that $(\mathcal{A}_i - \mathcal{L}_i C_i) \geq 0$ *if and only if* $(\mathcal{A}_i^T - C_i^T \mathcal{L}_i^T) \geq 0$*. Therefore, it follows that (39) is equivalent to the existence of gain matrices* $\mathcal{L}_i$ *that satisfy the condition* $(\mathcal{A}_i^T - C_i^T \mathcal{L}_i^T) \geq 0$*. Given that* $\underline{\nu}_i > 0$*, it follows that,* $(\mathcal{A}_i^T - C_i^T \mathcal{L}_i^T)\underline{\nu}_i \geq 0$*. From (52), we have*

$$\begin{aligned} a_{ql}^i \underline{\nu}_q \underline{\nu}_q^{-1} - c_l^{iT} \underline{\nu}_q^{-1} z_q^i = (A_i^T - C_i^T L_i^T)_{lq} \geq 0 \\ for \quad l, q = 1, \ldots, n \end{aligned} \tag{68}$$

Hence, (39) $\Rightarrow$ *(52). A similar development is used to obtain (28)* $\Rightarrow$ *(76) and (41)* $\Rightarrow$ *(54). This completes the proof.* □

Consider the system (1) with interval uncertainties expressed as

$$0 \leq \underline{A}_i \leq \mathcal{A}_i \leq \bar{A}_i, \quad 0 \leq \underline{A}_{\tau i} \leq \mathcal{A}_{\tau i} \leq \bar{A}_{\tau i}, \tag{69}$$

where $\underline{A}_i, \bar{A}_i, \underline{A}_{\tau i}, \bar{A}_{\tau i}$ are known matrices.
Recall that the problem addressed here is for given matrices $\underline{A}_i, \bar{A}_i, \underline{A}_{\tau i}, \bar{A}_{\tau i}$ find the gain matrices $\mathcal{L}_i$ that ensure the asymptotic stability and positivity of System (8) for any switching sequence σ_k satisfying ADT (16).
The following theorem proposes the conditions solving the problem addressed above in form of Linear Programming.

Theorem 4.2 *The solution of a mixed* $L\_/L_1$ *fault detection positive observer design is obtained for any switching sequence with ADT (16) satisfying (57-59), if there exist vectors* $\underline{\nu}_i > 0$, $\underline{\vartheta}_i > 0$, $\underline{\kappa}_i > 0$, $\underline{\eta}_i > 0$ *and* $z_1^i, \cdots, z_n^i \in \mathbb{R}^p$ *and for given scalars* $\underline{\beta} > 0$, $\underline{\gamma} > 0$,

$0 < \tilde{\zeta} < 1$ *and* $\tilde{\iota} \geq 1$ *such that*

$$(\bar{A}_i^T - \mathbb{I})\underline{\nu}_i + \tilde{\zeta}\underline{\nu}_i - C_i^T \sum_{l=1}^{n} z_l^i + C_i^T \boldsymbol{e} + (1-\tilde{\zeta})\underline{\eta}_i \quad < \quad 0 \tag{70}$$

$$F_i^T(\underline{\kappa}_i + \underline{\nu}_i) - M_i^T \sum_{l=1}^{n} z_l^i - M_i^T \boldsymbol{e} + \underline{\beta}\boldsymbol{e} \quad < \quad 0 \tag{71}$$

$$E_i^T(\underline{\kappa}_i + \underline{\nu}_i) - N_i^T \sum_{l=1}^{n} z_l^i + N_i^T \boldsymbol{e} + \underline{\gamma}\boldsymbol{e} \quad < \quad 0 \tag{72}$$

$$(\bar{A}_i^T - \mathbb{I})\underline{\kappa}_i + \tilde{\zeta}\underline{\kappa}_i + (1-\tilde{\zeta})\underline{\vartheta}_i \quad < \quad 0 \tag{73}$$

$$\bar{A}_{\tau i}^T\underline{\kappa}_i - (1-\tilde{\zeta})^{\tau+1}\underline{\vartheta}_i \quad < \quad 0 \tag{74}$$

$$\bar{A}_{\tau i}^T\underline{\nu}_i - (1-\tilde{\zeta})^{\tau+1}\underline{\eta}_i \quad < \quad 0 \tag{75}$$

$$\underline{a}_{ql}^i\underline{\nu}_q - c_l^{iT} z_q^i \quad \geq \quad 0 \tag{76}$$

$$e_{qs}^i\underline{\nu}_q - n_s^{iT} z_q^i \quad \geq \quad 0 \tag{77}$$

$$f_{qk}^i\underline{\nu}_q - m_k^{iT} z_q^i \quad \geq \quad 0 \tag{78}$$

$$c_l^{iT} z_q^i \quad \geq \quad 0 \tag{79}$$

$$z_l^i \quad > \quad 0 \tag{80}$$

Proof 7 *One has* $\underline{A}_i \leq \mathcal{A}_i \leq \bar{A}_i$, $\underline{A}_{\tau i} \leq \mathcal{A}_{\tau i} \leq \bar{A}_{\tau i}$. *As* $\underline{\nu}_i > 0$ *and* $\underline{\kappa}_i > 0$, *one can write* $\underline{\phi}_i \leq (A_i^T - \mathbb{I})\underline{\nu}_i + \tilde{\zeta}\underline{\nu}_i - C_i^T \sum_{l=1}^{n} z_l^i + (1-\tilde{\zeta})\underline{\eta}_i + C_i^T \boldsymbol{e} \leq \overline{\phi}_i, \underline{\psi}_i \leq (A_i^T + \tilde{\zeta} - \mathbb{I})\underline{\kappa}_i - (1-\tilde{\zeta})\underline{\vartheta}_i \leq \bar{\psi}_i, \underline{\Omega}_i \leq A_{\tau i}^T\underline{\kappa}_i - (1-\tilde{\zeta})^{\tau+1}\underline{\vartheta}_i \leq \overline{\Omega}_i, \underline{\Xi}_i \leq A_{\tau i}^T\underline{\nu}_i - (1-\tilde{\zeta})^{\tau+1}\underline{\eta}_i \leq \overline{\Xi}_i$.
with

$$\begin{aligned}
\underline{\phi}_i &= (\underline{A}_i^T - \mathbb{I})\underline{\nu}_i + \tilde{\zeta}\underline{\nu}_i - C_i^T \sum_{l=1}^{n} z_l^i + (1-\tilde{\zeta})\underline{\eta}_i + C_i^T \boldsymbol{e} \\
\overline{\phi}_i &= (\bar{A}_i^T - \mathbb{I})\underline{\nu}_i + \tilde{\zeta}\underline{\nu}_i - C_i^T \sum_{l=1}^{n} z_l^i + (1-\tilde{\zeta})\underline{\eta}_i + C_i^T \boldsymbol{e} \\
\underline{\psi}_i &= (\underline{A}_i^T + \tilde{\zeta} - \mathbb{I})\underline{\kappa}_i - (1-\tilde{\zeta})\underline{\vartheta}_i \\
\overline{\psi}_i &= (\bar{A}_i^T + \tilde{\zeta} - \mathbb{I})\underline{\kappa}_i - (1-\tilde{\zeta})\underline{\vartheta}_i \\
\underline{\Omega}_i &= \underline{A}_{\tau i}^T\underline{\kappa}_i - (1-\tilde{\zeta})^{\tau+1}\underline{\vartheta}_i \\
\overline{\Omega}_i &= \overline{A}_{\tau i}^T\underline{\kappa}_i - (1-\tilde{\zeta})^{\tau+1}\underline{\vartheta}_i \\
\underline{\Xi}_i &= \underline{A}_{\tau i}^T\underline{\nu}_i - (1-\tilde{\zeta})^{\tau+1}\underline{\eta}_i \\
\overline{\Xi}_i &= \overline{A}_{\tau i}^T\underline{\nu}_i - (1-\tilde{\zeta})^{\tau+1}\underline{\eta}_i
\end{aligned}$$

Therefore, $\overline{\phi}_i < 0$, $\overline{\Omega}_i < 0$, *and* $\overline{\Xi}_i < 0$ *are sufficient conditions for asymptotic stability of the augmented system (8). Similarly,* $\underline{a}_{ql}^i \leq a_{ql}^i \leq \bar{a}_{ql}^i$. *Since* $\underline{\nu}_q > 0$. *So,* $\underline{a}_{ql}^i\underline{\nu}_q - c_l^{iT} z_q^i \leq a_{ql}^i\underline{\nu}_q - c_l^{iT} z_q^i \leq \bar{a}_{ql}^i\underline{\nu}_q - c_l^{iT} z_q^i$. *Consequently,* $\underline{a}_{ql}^i\underline{\nu}_q - c_l^{iT} z_q^i \geq 0$ *is sufficient condition to ensure always that* $a_{ql}^i\underline{\nu}_q - c_l^{iT} z_q^i \geq 0$. *This completes the proof* □

5 Simulations

Consider the following uncertain interval switched system:

Subsystem 1:

$$\bar{A}_1 = \begin{bmatrix} 0.4 & 0.1 \\ 0.3 & 0.4 \end{bmatrix}, \underline{A}_1 = \begin{bmatrix} 0.26 & 0.01 \\ 0.2 & 0.2 \end{bmatrix}, \bar{A}_{\tau 1} = \begin{bmatrix} 0.2 & 0 \\ 0.2 & 0 \end{bmatrix}, \underline{A}_{\tau 1} = \begin{bmatrix} 0.08 & 0 \\ 0.08 & 0 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix}, F_1 = \begin{bmatrix} 0.3 \\ 0.32 \end{bmatrix}, N_1 = 0.1, C_1 = \begin{bmatrix} 0.12 & 0.13 \end{bmatrix}, M_1 = 0.12.$$

Subsystem 2:

$$\bar{A}_2 = \begin{bmatrix} 0.3 & 0.4 \\ 0.3 & 0.5 \end{bmatrix}, \underline{A}_2 = \begin{bmatrix} 0.16 & 0.25 \\ 0.18 & 0.3 \end{bmatrix}, \bar{A}_{\tau 2} = \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}, \underline{A}_{\tau 2} = \begin{bmatrix} 0.01 & 0 \\ 0.01 & 0 \end{bmatrix},$$

$$E_2 = \begin{bmatrix} 0.32 \\ 0.28 \end{bmatrix}, F_2 = \begin{bmatrix} 0.28 \\ 0.3 \end{bmatrix}, N_2 = 0.1, C_2 = \begin{bmatrix} 0.12 & 0.15 \end{bmatrix}, M_2 = 0.12, \tau = 3.$$

Set $\underline{\gamma} = 0.9$, $\underline{\beta} = 0.009$. From Theorem 4.2 and for any switching signal σ_k with ADT parameters chosen as $\tilde{\iota} = 1.3$ and $\tilde{\zeta} = 0.2$, which implies $\tau_a \geq \tau_a^* = 1.1758$. The gains matrices $\mathcal{L}_i$ are designed as follows: $L_1 = \begin{bmatrix} 0.2916 \\ 1.5058 \end{bmatrix}, L_2 = \begin{bmatrix} 1.0543 \\ 0.9746 \end{bmatrix}$. The minimal and the maximal positivity conditions are given by:

$$\bar{A}_1 - L_1 C_1 = \begin{bmatrix} 0.3650 & 0.0621 \\ 0.1193 & 0.2042 \end{bmatrix}, \underline{A}_1 - L_1 C_1 = \begin{bmatrix} 0.1735 & 0.2419 \\ 0.1830 & 0.3538 \end{bmatrix},$$

$$\bar{A}_2 - L_2 C_2 = \begin{bmatrix} 0.2250 & 0.0621 \\ 0.0193 & 0.0042 \end{bmatrix}, \underline{A}_2 - L_2 C_2 = \begin{bmatrix} 0.0335 & 0.0919 \\ 0.0630 & 0.1538 \end{bmatrix},$$

The switching sequences is illustrated in Figure 1 while the residual criterian is presented

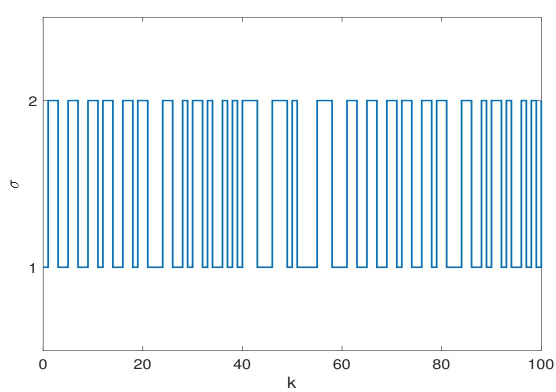

Figure 1: Evolution of the switching signal σ_k.

in Figure 2 showing the exact interval time of the presence of the fault. The response of the evaluation function J_k is given in Figure 3. This function gives information about occurrence and duration of the fault. In addition, Figure 4 and 5 show the evolution of the uncertain states. It is important to mention that the values of the uncertainties are taken variable during the simulation time. The uncertain system trajectory is asymptotically stable and positive even in the presence of a fault occurring at $k = 50$.

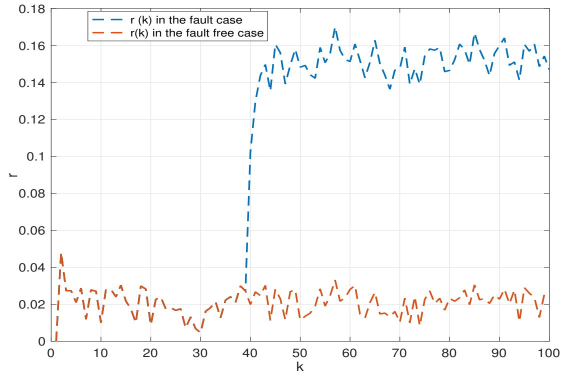

Figure 2: Evolution of the residual r_k.

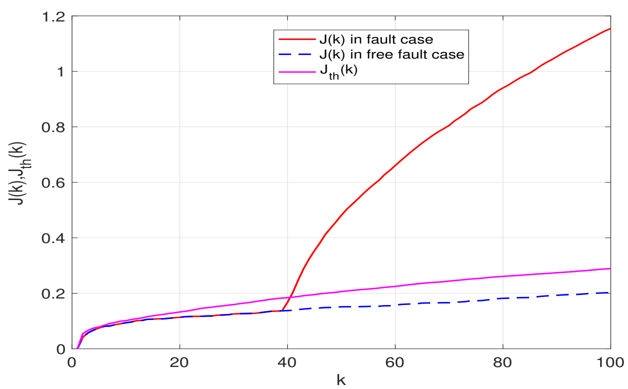

Figure 3: Response of the evaluation function J_k with free fault and fault case.

6 CONCLUSION

This paper is devoted to investigate the issue of robust FD with time delays and interval uncertainties. By using the multiple Lyapunov function and the ADT constraints. Hence, the designed observers gains satisfy the stability and positivity of the augmented system while the robustness conditions are respected. Finally, a numerical example is presented to prove the effectiveness of the developed results. As perspectives of this paper, we believe that an extension to fault detection based on neural network observer can be done using the same technique developed in [12].

Acknowledgment

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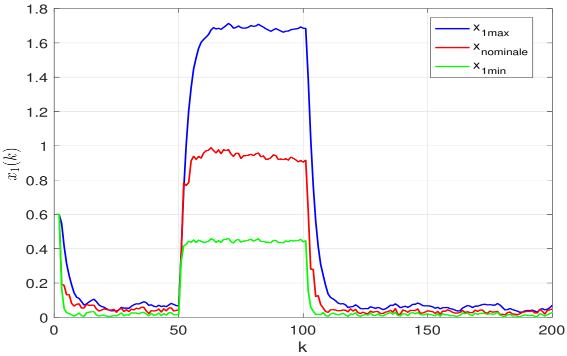

Figure 4: state $x_1(k)$ with time-varying uncertainty.

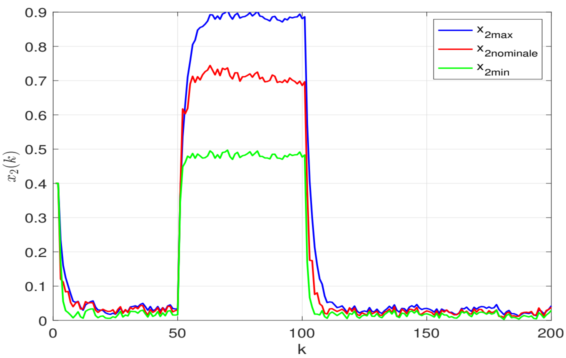

Figure 5: state $x_2(k)$ with time-varying uncertainty.

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