# Data Communication Network Based Active Fault-Tolerant Control of Positive Switched Systems

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**Abstract.** This paper investigates Active Fault Tolerant Control (AFTC) strategies for positive switched systems under Average Dwell Time (ADT) constraints. Employing multiple Lyapunov functions, sufficient conditions are derived. Leveraging these results, two theorems are developed to ensure asymptotic stability of the closed-loop system, accounting for the effects of unknown input. To conclude, a practical example involving a data communication network is presented to demonstrate its effectiveness.

## 1 Introduction

A positive system is a class of dynamical systems in which the state variables and outputs remain non-negative for all time, given non-negative initial conditions and inputs. These systems are frequently used in real-world applications such as energy systems, biological systems, and data communication networks [1–3]. Notably, positive switched systems are particularly challenging, as they require the simultaneous consideration of both positivity constraints and the complexities introduced by switching mechanisms. Lately, positive switched systems have emerged as a prominent topic in control theory, with ongoing research addressing key issues such as stability and stabilization [4–6].

The increasing demand for safety and reliability in modern engineering applications has positioned FTC as a crucial control strategy in recent years [7–9]. FTC plays a vital role in maintaining system performance and stability even in the presence of faults. FTC approaches are commonly categorized into two main types: active fault-tolerant control (AFTC) and passive fault-tolerant control (PFTC). AFTC relies on fault estimation (FE) mechanisms combined with reconfiguration or accommodation strategies to counteract faults effectively. In contrast, PFTC designs robust control laws capable of preserving acceptable performance even in the presence of time- varying uncertainties or faults. In comparison, AFTC requires real-time fault estimation in order to guarantee the continued operation through the control compensation techniques while PFTC designs a robust control law under a set of assumed fault conditions. Numerous control methodologies have been applied within the FTC framework, such as state feedback control [10], sliding mode control [11], adaptive control methods [12], and neural network-based strategies [13], etc.

It is well known that ADT approach plays a crucial role in ensuring stability in positive switched systems. ADT defines a bound on the number of switches within a finite time

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interval to maintain system stability and performance. This concept is increasingly relevant in data communication networks where frequent switching such as between routes, channels, or access points can lead to instability, increased overhead, and degraded quality of service. By applying the ADT method, the number of switches is regulated over a finite time interval, ensuring that the system spends sufficient time in each subsystem before switching occurs. This constraint is essential in preserving the positivity and fault tolerant performance, while also maintaining global asymptotic stability.

Motivated by these advancements, this paper considers an active fault tolerant control while considering ADT and positivity constraints for linear switched systems. The contribution of this work can be summarized as follows:

- Our approach employs a PI observer, which yields less conservative results compared to the method presented in [14], which uses a Luenberger observer.
- Compared to the work presented in [7], our method incorporates ADT constraints, which play a crucial role in enhancing the control mechanism by ensuring system stability, consistent performance, and efficient switching behavior.
- Most recent works on positive switched systems, such as [7] use numerical examples to demonstrate the effectiveness of their results. In contrast, this paper proposes a different approach. The proposed method is applied to fault estimation and accommodation in a data communication network with three nodes under an active network attack.
- Unknown inputs are a critical factor in control systems. Unlike the work presented in [14], which does not consider its impact, our approach explicitly accounts for disturbances and addresses them using the $\mathcal{H}_\infty$ method. This allows us to effectively minimize their influence on system performance, thereby enhancing robustness and reliability.

Throughout this paper, the following notations are adopted:

- For a matrix $M$, $M^T$ denotes its transpose.
- The symbol * will be used to express the symmetric terms.
- 0 and $\mathbb{I}$ denotes the null and identity matrix with appropriate size, respectively.

## 2 Problem statement and some preliminaries results

Consider the following positive switched system:

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}_{\sigma(t)}x(t) + \mathcal{B}_{\sigma(t)}(u(t) + f(t)) + \mathcal{E}_{\sigma(t)}d(t) \\ y(t) &= C_{\sigma(t)}x(t) \\ \tilde{x}(t_o) &= x_o \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $f(t) \in \mathbb{R}^m$ , $d(t) \in \mathbb{R}^g$ and $y(t) \in \mathbb{R}^p$, are respectively, the state, the control input, the fault, the bounded arbitrary disturbance input and the output. We suppose the switching signal $\sigma(t)$ time-dependent, i.e. $\sigma(t) : \{(t_0, \sigma(t_0)), (t_1, \sigma(t_1)), \cdots (t_k, \sigma(t_k)), \cdots, | \; k = 0, 1, \cdots\}$. Some preliminaries are recalled for the convenes of later proofs before presenting the next section.

**Definition 2.1** *[15] If for any non-negative initial conditions $\tilde{x}(t_0) \geq 0$, non-negative unknown inputs $d_t \geq 0$, any fault $f_t \geq 0$, the corresponding trajectories $x_t \geq 0$ and $y_t$ remain non-negative for all $t \geq 0$.*

**Definition 2.2** *[16] A non positive system is said to be controlled positive, if the system in closed-loop is forced by the controller to be positive.*

**Definition 2.3** *[17] $A_i$ is said Metzler matrix if its off diagonal elements are non negative, that is $a_{ij} \leq 0$, for $i \neq j$.*

**Lemma 2.1** *[17] If and only if $\mathcal{A}_i \geq 0$, $C_i \geq 0$ and $\mathcal{E}_i \geq 0$, then, system (1) is positive.*

**Definition 2.4** *[17] For any switching signal $\sigma(t)$ and any $t_2 > t_1 > 0$, let $\mathcal{N}_\sigma(t_1, t_2)$ denotes the number of switching of $\sigma(t)$, on an interval $(t_1, t_2)$. If*

$$\mathcal{N}_\sigma(t_1, t_2) \leq \mathcal{N}_0 + \frac{t_2 - t_1}{\tau_a}$$

*holds for a given $\mathcal{N}_0 \geq 0$ and $\tau_a$, then the constant $\tau_a$ is called the average dwell time ADT and $\mathcal{N}_0 \geq 0$ is called the chattering bound.*

**Lemma 2.2** *[18] Given symmetric matrix $\tilde{\mathcal{S}}$ and matrices $\tilde{\mathcal{T}}, \tilde{\Sigma}(t)$ and $\tilde{\mathcal{H}}$ of appropriate size, then*

$$\tilde{\mathcal{S}} + \tilde{\mathcal{T}}^T \tilde{\Sigma}(t) \tilde{\mathcal{H}} + \tilde{\mathcal{H}}^T \tilde{\Sigma}(t)^T \tilde{\mathcal{T}} < 0,$$

*holds for $\tilde{\Sigma}(t)^T \tilde{\Sigma}(t) \leq \mathbb{I}$ if and only if there exists a positive scalar $\epsilon$ such that*

$$\tilde{\mathcal{S}} + \epsilon^{-1} \tilde{\mathcal{T}}^T \tilde{\mathcal{T}} + \epsilon \tilde{\mathcal{H}}^T \tilde{\mathcal{H}} < 0.$$

**Lemma 2.3** *[17] If a real scale function $\phi(t)$ satisfies the following differential inequality:*

$$\dot{\phi}(t) \leq -\zeta \phi(t) + \varpi \delta(t) \tag{2}$$

*where $\zeta > 0$ and $\varpi > 0$, then*

$$\phi(t) \leq e^{-\zeta t} \phi(0) + \varpi \int_0^t e^{-\zeta \tau} \delta(t - \tau) \mathrm{d}\tau \tag{3}$$

## 3 Main results

The purpose of this section is to design an AFTC approach for positive switched systems while taking into account the positivity and ADT constraints.

### 3.1 Robust fault estimation design

To estimate constant or time varying faults, the following SPIO is given:

$$\begin{aligned} \dot{\hat{x}}(t) &= \mathcal{A}_i \hat{x}(t) + \mathcal{B}_i (u(t) + \hat{f}(t)) + \mathcal{L}_i (y(t) - \hat{y}(t)) \\ \hat{y}(t) &= C_i \hat{x}(t) \\ \dot{\hat{f}}(t) &= \Lambda_i [\mathcal{K}_i^P C_i \dot{\xi}(t) + \mathcal{K}_i^I C_i \xi(t)] \end{aligned} \tag{4}$$

where $\mathcal{K}_i^P$ and $\mathcal{K}_i^I$ are the proportional and integral gains, respectively, $\mathcal{L}_i$ is the observer gain and the symmetric positive definite matrix $\Lambda_i$ represents the learning rate.

Setting $\dot{\xi}(t) = \dot{x}(t) - \dot{\hat{x}}(t)$, let

$$\dot{\xi}(t) = (\mathcal{A}_i - \mathcal{L}_i C_i) \xi(t) + \mathcal{B}_i \xi_f(t) + \mathcal{E}_i d(t)$$

Defining $\dot{\xi}_f(t) = \dot{f}(t) - \dot{\hat{f}}(t)$, one get

$$\dot{\xi}_f(t) = \dot{f}(t) - \Lambda_i [(\mathcal{K}_i^P C_i \mathcal{A}_i + \mathcal{K}_i^I C_i - \mathcal{K}_i^P C_i \mathcal{L}_i C_i) \xi(t) + \mathcal{K}_i^P C_i \mathcal{B}_i \xi_f(t) - \mathcal{K}_i^P C_i \mathcal{E}_i d(t)]$$

The vectors of the augmented state and input are given by $\tilde{\xi}(t) = \begin{bmatrix} \xi(t) \\ \xi_f(t) \end{bmatrix}, \tilde{\nu}(t) = \begin{bmatrix} d(t) \\ \dot{f}(t) \end{bmatrix}$. Then,

$$\dot{\tilde{\xi}}(t) = \tilde{\mathcal{A}}_i\tilde{\xi}(t) + \tilde{\mathcal{D}}_i\tilde{\nu}(t) \tag{5}$$

where

$$\tilde{\mathcal{A}}_i = \begin{bmatrix} (\mathcal{A}_i - \mathcal{L}_i\mathcal{C}_i) & \mathcal{B}_i \\ \tilde{\Theta}_{ai} & \tilde{\Theta}_{ci} \end{bmatrix}, \tilde{\mathcal{D}}_i = \begin{bmatrix} \mathcal{E}_i & 0 \\ \tilde{\Theta}_{ni} & \mathbb{I} \end{bmatrix},$$

$$\tilde{\Theta}_{ai} = -\Lambda_i(\mathcal{K}_i^I\mathcal{C}_i + \mathcal{K}_i^P\mathcal{C}_i\mathcal{A}_i - \mathcal{K}_i^P\mathcal{C}_i\mathcal{L}_i\mathcal{C}_i), \tilde{\Theta}_{ci} = -\Lambda_i\mathcal{K}_i^P\mathcal{C}_i\mathcal{B}_i, \tilde{\Theta}_{ni} = -\Lambda_i\mathcal{K}_i^P\mathcal{C}_i\mathcal{E}_i.$$

The weighted $\mathcal{H}_\infty$ technique with respect to the ADT constraint is used.

$$\int_0^\infty e^{-\alpha s}\tilde{\xi}^T(s)\tilde{\mathcal{W}}\tilde{\xi}(s)ds - \gamma_1^2\int_0^\infty \tilde{\nu}^T(s)\tilde{\nu}(s)ds \leq 0, \tag{6}$$

with

$$\tilde{\mathcal{W}} = \begin{bmatrix} \mathcal{W}_1 & 0 \\ 0 & \mathcal{W}_2 \end{bmatrix}. \tag{7}$$

The following result design a robust actuator FE algorithm.

**Theorem 3.1** *Given constants $\alpha$, $\gamma_1, \epsilon_i$, and $\mu \geq 1$, the augmented estimation error (5) under switching signal with average dwell time $\tau_a$ is asymptotically stable with weighted $\mathcal{H}_\infty$ performance, if*

$$\begin{bmatrix} -\tilde{\Xi}_{1i} & \tilde{\Xi}_{2i} & \tilde{\Xi}_{4i} & 0 & 0 & \tilde{\Xi}_{11i} \\ * & -\tilde{\Xi}_{5i} & \tilde{\Xi}_{7i} & \Upsilon_i & \tilde{\Xi}_{10i} & 0 \\ * & * & -\gamma_1^2\mathbb{I} & 0 & 0 & 0 \\ * & * & * & -\gamma_1^2\mathbb{I} & 0 & 0 \\ * & * & * & * & -\epsilon_i^{-1}\mathbb{I} & 0 \\ * & * & * & * & * & -\tilde{\Xi}_{12i} \end{bmatrix} < 0 \tag{8}$$

$$\mathcal{P}_i \leq \mu\mathcal{P}_j \tag{9}$$

*and the ADT satisfying*

$$\tau_a > \tau_a^* = \frac{\ln\mu}{\alpha} \tag{10}$$

*where*

$$\tilde{\Xi}_{11i} = (\mathbb{Z}_i\mathcal{C}_i)^T \text{ and } \tilde{\Xi}_{12i} = 2\kappa\mathcal{P}_i - \kappa^2\epsilon_i^{-1}\mathbb{I}.$$

**Proof 1** *Consider the following multiple Lyapunov function with $\tilde{\nu}(t) = 0$:*

$$\mathcal{V}_i(t) = \tilde{\xi}^T(t)\,\tilde{\mathcal{P}}_i\,\tilde{\xi}(t) \tag{11}$$

*where*

$$\tilde{\mathcal{P}}_i = \begin{bmatrix} \mathcal{P}_i & 0 \\ 0 & \tilde{\Lambda}_i^{-1} \end{bmatrix}$$

*Along the trajectory of the augmented estimation error in (5), we have*

$$\dot{\mathcal{V}}_i(t) + \alpha\mathcal{V}_i(t) = \dot{\tilde{\xi}}^T(t)\tilde{\mathcal{P}}_i\tilde{\xi}(t) + \tilde{\xi}^T(t)\tilde{\mathcal{P}}_i\dot{\tilde{\xi}}(t) + \alpha\tilde{\xi}^T(t)\tilde{\mathcal{P}}_i\tilde{\xi}(t) \tag{12}$$

*Substituting* (5) *into (12), we obtain*

$$\dot{\mathcal{V}}_i(t) + \alpha\mathcal{V}_i(t) = \tilde{\xi}^T(t)(\tilde{\mathcal{A}}_i^T\tilde{\mathcal{P}}_i + \bar{\mathcal{P}}_i\tilde{\mathcal{A}}_i + \alpha\tilde{\mathcal{P}}_i)\tilde{\xi}(t) \tag{13}$$

*Then by (20), we get*

$$\dot{\mathcal{V}}_i(t) + \alpha\mathcal{V}_i(t) \leq 0, \tag{14}$$

*it follows that*

$$\dot{\mathcal{V}}_i(t) \leq -\alpha\mathcal{V}_i(t) \tag{15}$$

*Let $t_0 < t_1 < t_2 < \ldots, t_k < t_{k+1} \ldots < t$ denote the switching instants of $\sigma(t)$ over the interval $[t_0, t]$. Then, for any $t \in [t_k, t_{k+1})$, it holds that*

$$\mathcal{V}_i\left(\tilde{\xi}(t)\right) \leq \mathcal{V}_i\left(\tilde{\xi}(t_k)\right)e^{-\alpha(t-t_k)} \tag{16}$$

*According to* (9), (16) *and over the interval $[t_0, t)$, we have*

$$\begin{aligned}\mathcal{V}_i(\tilde{\xi}(t)) &\leq \mu e^{-\alpha(t-t_k)}\mathcal{V}_{i-1}\left(\tilde{\xi}\left(t_k^-\right)\right)\\ &\leq \mu e^{-\alpha(t-t_{k-1})}\mathcal{V}_{i-1}\left(\tilde{\xi}\left(t_{k-1}\right)\right)\\ &\leq \mu^2 e^{-\alpha(t-t_{k-1})}\mathcal{V}_{i-2}\left(\tilde{\xi}\left(t_{k-1}^-\right)\right)\\ &\vdots\\ &\leq \mu^{\mathcal{N}_\sigma(t_0,t)}e^{-\alpha(t-t_0)}\mathcal{V}_0\left(\tilde{\xi}\left(t_0\right)\right)\end{aligned} \tag{17}$$

*Using, $\mathcal{N}_\sigma(t_0, t) \leq \mathcal{N}_0 + (t - t_0)/\tau_a$, (17), we get:*

$$\mathcal{V}_i(\tilde{\xi}(t)) \leq \mu^{\mathcal{N}_0}e^{-(\alpha-\ln(\mu)/\tau_a)(t-t_0)}\mathcal{V}_0\left(\tilde{\xi}\left(t_0\right)\right)$$

*If the ADT satisfies* (10)*, then, $\mathcal{V}_i(t)$ converges exponentially to zero when $t \to \infty$. As a result, (5) is asymptotic stable.*
*Now, the proof of the weighted $\mathcal{H}_\infty$ performance is considered.*
*Let $\tilde{J}_1(t) = -\tilde{\xi}^T(t)\tilde{\mathcal{W}}\tilde{\xi}(t) + \gamma_1^2\tilde{\nu}(t)^T\tilde{\nu}(t)$, one obtain*

$$\dot{\mathcal{V}}_i(t) + \alpha\mathcal{V}_i(t) - \tilde{J}1(t) \leq \beta^T(t)\Gamma_i\beta(t) \tag{18}$$

*where*

$$\Gamma_i = \begin{bmatrix} \tilde{\mathcal{A}}_i^T\tilde{\mathcal{P}}_i + \tilde{\mathcal{P}}_i\tilde{\mathcal{A}}_i + \alpha\tilde{\mathcal{P}}_i + \tilde{\mathcal{W}} & \tilde{\mathcal{P}}_i\tilde{\mathcal{D}}_i \\ * & -\gamma_1^2\mathbb{I} \end{bmatrix}, \beta(t) = \begin{bmatrix} \tilde{\xi}(t) \\ \tilde{\nu}(t) \end{bmatrix}$$

*Substituting the corresponding values of $\tilde{\mathcal{P}}_i$, $\tilde{\mathcal{A}}_i$ and $\tilde{\mathcal{D}}_i$ into $\Gamma_i$, one has*

$$\Gamma_i = \begin{bmatrix} -\tilde{\Xi}_{1i} & \Xi_{2i} & \tilde{\Xi}_{4i} & 0 \\ * & -\tilde{\Xi}_{5i} & \Xi_{7i} & \Upsilon_i \\ * & * & -\gamma_1^2\mathbb{I} & 0 \\ * & * & * & -\gamma_1^2\mathbb{I} \end{bmatrix}$$

*where $\tilde{\Xi}_{2i} = -(\mathcal{K}_i^P C_i\mathcal{A}_i)^T - (\mathcal{K}_i^I C_i)^T + \mathcal{P}_i\mathcal{B}_i + (\mathcal{K}_i^P C_i\mathcal{L}_i C_i)^T$. Matrix $\Gamma_i$ is rewritten as follows:*

$$\Gamma_i = \Gamma_{oi} + \tilde{\mathcal{T}}_i^T\tilde{\mathcal{H}}_i + \tilde{\mathcal{H}}_i^T\tilde{\mathcal{T}}_i, \tag{19}$$

*where*

$$\Gamma_{oi} = \begin{bmatrix} -\tilde{\Xi}_{1i} & \tilde{\Xi}_{2i} & \tilde{\Xi}_{4i} & 0 \\ * & -\tilde{\Xi}_{5i} & \tilde{\Xi}_{7i} & \Upsilon_i \\ * & * & -\gamma_1^2\mathbb{I} & 0 \\ * & * & * & -\gamma_1^2\mathbb{I} \end{bmatrix}, \tilde{\mathcal{T}}_i = [\mathcal{L}_i C_i \quad 0 \quad 0 \quad 0], \quad \tilde{\mathcal{H}}_i = [0 \quad (\mathcal{K}_i^I C_i)^T \quad 0 \quad 0].$$

*Using Lemma 2.2 and Schur complement as* $\tilde{\Sigma} = \mathbb{I}$*, we obtain the following Inequality:*

$$\Xi_i = \begin{bmatrix} -\tilde{\Xi}_{1i} & \tilde{\Xi}_{2i} & \tilde{\Xi}_{4i} & 0 & 0 & \tilde{\Xi}_{11i} \\ * & -\tilde{\Xi}_{5i} & \tilde{\Xi}_{7i} & \Upsilon_i & \tilde{\Xi}_{10i} & 0 \\ * & * & -\gamma_1^2\mathbb{I} & 0 & 0 & 0 \\ * & * & * & -\gamma_1^2\mathbb{I} & 0 & 0 \\ * & * & * & * & -\epsilon_i^{-1}\mathbb{I} & 0 \\ * & * & * & * & * & -\epsilon_i\mathbb{I} \end{bmatrix} < 0 \tag{20}$$

*where*

$$\begin{aligned} \tilde{\Xi}_{1i} &= -\mathcal{A}_i^T\mathcal{P}_i - \mathcal{P}_i\mathcal{A}_i + (\mathbb{Z}_i C_i)^T + (\mathbb{Z}_i C_i) - \alpha\mathcal{P}_i - \mathcal{W}_1, \\ \tilde{\Xi}_{2i} &= -(\mathcal{K}_i^P C_i\mathcal{A}_i)^T - (\mathcal{K}_i^I C_i)^T + \mathcal{P}_i\mathcal{B}_i, \Upsilon_i = \Lambda_i^{-1}, \\ \tilde{\Xi}_{4i} &= \mathcal{P}_i\mathcal{E}_i, \ \tilde{\Xi}_{5i} = (\mathcal{K}_i^P C_i\mathcal{B}_i) + (\mathcal{K}_i^P C_i\mathcal{B}_i)^T - \mathcal{W}_2, \tilde{\Xi}_{7i} = -\mathcal{K}_i^P C_i\mathcal{E}_i, \tilde{\Xi}_{10i} = \mathcal{K}_i^P C_i, \\ \tilde{\Xi}_{11i} &= (\mathcal{L}_i C_i)^T, \ \mathbb{Z}_i = \mathcal{P}_i\mathcal{L}_i \ . \end{aligned} \tag{21}$$

*Therefore, if* (20) *is satisfied, then,* $\Gamma_i \leq 0$*. It is clear that*

$$\dot{\mathcal{V}}_i(t) \leq -\alpha\mathcal{V}_i(t) + \tilde{J}_1(t) \tag{22}$$

*By Lemma 2.3, one has*

$$\mathcal{V}_i(t) \leq e^{-\alpha(t-t_k)}\mathcal{V}_i(t_k) + \int_{t_k}^{t} e^{-\alpha(t-s)}\tilde{J}_1(s)\mathrm{d}s \tag{23}$$

*Denote* $t_0, t_1, \cdots t_k, t_{k+1} \cdots$ *as the switching instants during interval* $[t_0, t]$ *and according to* (23) *and* (9)*, we have*

$$\begin{aligned} \mathcal{V}_i(t) &\leq e^{-\alpha(t-t_k)}\mathcal{V}_i(t_k) + \int_{t_k}^{t} e^{-\alpha(t-s)}\tilde{J}_1(s)\mathrm{d}s \\ &\leq \mu e^{-\alpha(t-t_k)}\mathcal{V}_{i-1}\left(t_k^-\right) + \int_{t_k}^{t} e^{-\alpha(t-s)}\tilde{J}_1(s)\mathrm{d}s \\ &\leq \mu e^{-\alpha(t-t_{k-1})}\mathcal{V}_{i-1}(t_{k-1}) + \mu\int_{t_{k-1}}^{t_k} e^{-\alpha(t-s)}\tilde{J}_1(s)\mathrm{d}s + \int_{t_k}^{t} e^{-\alpha(t-s)}\tilde{J}_1(s)\mathrm{d}s \\ &\leq \mu^2 e^{-\alpha(t-t_{k-1})}\mathcal{V}_{i-2}\left(t_{k-1}^-\right) + \mu\int_{t_{k-1}}^{t_k} e^{-\alpha(t-s)}\tilde{J}_1(s)\mathrm{d}s + \int_{t_k}^{t} e^{-\alpha(t-s)}\tilde{J}_1(s)\mathrm{d}s \\ &\leq \ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \end{aligned} \tag{24}$$

$$\begin{aligned} &\leq \mu^{\mathcal{N}_\sigma(t_0,t)} e^{-\alpha(t-t_0)}\mathcal{V}_0(t_0) + \mu^{\mathcal{N}_\sigma(t_0,t)}\int_{t_0}^{t_1} e^{-\alpha(t-s)}\tilde{J}_1(s)\mathrm{d}s + \cdots \\ &+ \mu^{\mathcal{N}_\sigma(t_{k-1},t)}\int_{t_{k-1}}^{t_k} e^{-\alpha(t-s)}\tilde{J}_1(s)\mathrm{d}s + \mu^{\mathcal{N}_\sigma(t_k,t)}\int_{t_k}^{t} e^{-\alpha(t-s)}\tilde{J}_1(s)\mathrm{d}s \\ &= \mu^{\mathcal{N}_\sigma(t_0,t)} e^{-\alpha(t-t_0)}\mathcal{V}_0(t_0) + \int_{t_0}^{t}\mu^{\mathcal{N}_\sigma(s,t)} e^{-\alpha(t-s)}\tilde{J}_1(s)\mathrm{d}s \end{aligned}$$

*Under zero initial condition,* (24) *implies*

$$\int_{t_0}^{t} e^{-\alpha(t-s)+\mathcal{N}_\sigma(s,t)\ln\mu}\tilde{\xi}^T(s)\tilde{\mathcal{W}}\tilde{\xi}(s)\mathrm{d}s \le \gamma_1^2\int_{t_0}^{t} e^{-\alpha(t-s)+\mathcal{N}_\sigma(s,t)\ln\mu}\tilde{\nu}^T(s)\tilde{\nu}(s)\mathrm{d}s. \tag{25}$$

*Multiplying both sides of (25) by* $e^{-\mathcal{N}_\sigma(0,t)\ln\mu}$ *yields*

$$\int_{t_0}^{t} e^{-\alpha(t-s)-\mathcal{N}_\sigma(0,s)\ln\mu}\tilde{\xi}^T(s)\tilde{\mathcal{W}}\tilde{\xi}(s)\mathrm{d}s \le \gamma_1^2\int_{t_0}^{t} e^{-\alpha(t-s)-\mathcal{N}_\sigma(0,s)\ln\mu}\tilde{\nu}^T(s)\tilde{\nu}(s)\mathrm{d}s \tag{26}$$

*Using* $\mathcal{N}_\sigma(0,s) \le s/\tau_a$ *and* $\tau_a \ge \frac{\ln\mu}{\alpha}$*; we have* $\mathcal{N}_\sigma(0,s)\ln\mu \le \alpha s$*. Hence,* (26) *implies*

$$\int_{t_0}^{t} e^{-\alpha(t-s)-\alpha s}\tilde{\xi}^T(s)\tilde{\mathcal{W}}\tilde{\xi}(s)\mathrm{d}s \le \gamma_1^2\int_{t_0}^{t} e^{-\alpha(t-s)}\tilde{\nu}^T(s)\tilde{\nu}(s)\mathrm{d}s \tag{27}$$

*For* $t_0 = 0$ *and* $t \to \infty$*, we get* (6)*.*

To highlight the estimator gain $\mathcal{L}_i$ within the component $\tilde{\Xi}_{11i}$, it is recommended to represent the matrix $\tilde{\Xi}_i$ in a form that incorporates the variable $\mathbb{Z}_i = \mathcal{P}_i\mathcal{L}_i$.
we have

$$\tilde{\Xi}_i = \begin{bmatrix} \bar{\Psi}_{11i} & \bar{\Psi}_{12i} \\ * & \bar{\Psi}_{22i} \end{bmatrix} < 0, \tag{28}$$

where

$$\bar{\Psi}_{11i} = \begin{bmatrix} -\tilde{\Xi}_{1i} & \tilde{\Xi}_{2i} & \tilde{\Xi}_{4i} & 0 & 0 \\ * & -\tilde{\Xi}_{5i} & \tilde{\Xi}_{7i} & \Upsilon_i & \tilde{\Xi}_{10i} \\ * & * & -\gamma_1^2\mathbb{I} & 0 & 0 \\ * & * & * & -\gamma_1^2\mathbb{I} & 0 \\ * & * & * & * & -\epsilon_i^{-1}\mathbb{I} \end{bmatrix}, \bar{\Psi}_{22i} = -\epsilon_i\mathbb{I}, \bar{\Psi}_{12i} = [\tilde{\Xi}_{11i}^T \quad 0 \quad 0 \quad 0 \quad 0]^T.$$

Pre and post multiplying Inequality (28) by $\Omega_i$, one can get

$$\begin{bmatrix} \bar{\Psi}_{11i} & \bar{\Psi}_{12i}\mathcal{P}_i \\ * & \mathcal{P}_i\bar{\Psi}_{22i}\mathcal{P}_i \end{bmatrix} < 0, \tag{29}$$

where

$$\Omega_i = \begin{bmatrix} \mathbb{I} & 0 \\ 0 & \mathcal{P}_i \end{bmatrix}.$$

Moreover, Inequality (28) implies that

$$\bar{\Psi}_{22i} < 0.$$

According to

$$(\mathcal{P}_i + \kappa\bar{\Psi}_{22i}^{-1})^T\bar{\Psi}_{22i}(\mathcal{P}_i + \kappa\bar{\Psi}_{22i}^{-1}) < 0.$$

we get

$$\mathcal{P}_i\bar{\Psi}_{22i}\mathcal{P}_i < -2\kappa\mathcal{P}_i - \kappa^2\bar{\Psi}_{22i}^{-1}, \quad \kappa \ge 1 \tag{30}$$

Replacing (30) into (29), one has

$$\begin{bmatrix} \bar{\Psi}_{11i} & \bar{\Psi}_{12i}\mathcal{P}_i \\ * & -2\kappa\mathcal{P}_i - \kappa^2\bar{\Psi}_{22i}^{-1} \end{bmatrix} < 0. \tag{31}$$

Replacing $\bar{\Psi}_{11i}, \bar{\Psi}_{22i}, \bar{\Psi}_{22i}$ into (31), the LMI (8) is obtained.

### 3.2 Fault accommodation design

Based on the obtained state and fault estimation information, an active fault tolerant controller is provided.

$$u(t) = \mathcal{G}_i \hat{x}(t) - \hat{f}(t) \tag{32}$$

where $\mathcal{G}_i$ are the controller gains. Substituting (32) into (1), we get

$$\begin{aligned} \dot{x}(t) &= \mathcal{A}_{ci} x(t) + \rho(t) \\ y(t) &= C_i x(t) \end{aligned} \tag{33}$$

where $\mathcal{A}_{ci} = \mathcal{A}_i + \mathcal{B}_i \mathcal{G}_i$, $\rho(t) = \delta(t) + \mathcal{E}_i \nu(t)$, $\delta(t) = -\mathcal{B}_i \mathcal{G}_i \xi(t) + B_i \xi_f(t)$. Introducing:

$$\int_0^\infty e^{-\alpha s} x^T(s) \mathcal{W} x(s) ds - \gamma_2^2 \int_0^\infty \rho^T(s) \rho(s) ds \leq 0 \tag{34}$$

**Theorem 3.2** *Given constants $\gamma_2$, $\alpha$ and $\mu \geq 1$ and if there exist a symmetric positive definite matrix $\mathcal{X}_i$ and matrices $\mathcal{Y}_i$, $\mathcal{G}_i$, such that the following LMI holds*

$$\begin{bmatrix} \mathcal{A}_i \mathcal{X}_i + \mathcal{B}_i \mathcal{Y}_i + \mathcal{X}_i \mathcal{A}_i^T + \mathcal{Y}_i^T \mathcal{B}_i^T + \alpha \mathcal{X}_i & \mathcal{E}_i & \mathcal{X}_i \\ * & -\gamma_2^2 \mathbb{I} & 0 \\ * & * & -\mathcal{W} \end{bmatrix} < 0 \tag{35}$$

$$\mathcal{X}_i \leq \mu \mathcal{X}_j$$

$$\mathcal{A}_i X_i + \mathcal{B}_i Y_i \quad \textit{are Metzler} \tag{36}$$

where $\mathcal{G}_i = \mathcal{Y}_i \mathcal{X}_i^{-1}$, then, for the switching signal $\sigma(t)$ with ADT constraint (10), System (33) is asymptotically stable.

**Proof 2** *Let us set $\rho(t) = 0$ and consider a multiple Lyapunov function as:*

$$\bar{\mathcal{V}}_i(t) = x^T(t) \bar{\mathcal{P}}_i x(t) \tag{37}$$

*One has*

$$\dot{\bar{\mathcal{V}}}_i(t) + \alpha \bar{\mathcal{V}}_i(t) = x^T(t) (\mathcal{A}_{ci}^T \bar{\mathcal{P}}_i + \bar{\mathcal{P}}_i \mathcal{A}_{ci} + \alpha \bar{\mathcal{P}}_i) x(t) \tag{38}$$

*then by (35), we get*

$$\dot{\bar{\mathcal{V}}}_i(t) + \alpha \bar{\mathcal{V}}_i(t) \leq 0, \tag{39}$$

*it follows that*

$$\dot{\bar{\mathcal{V}}}_i(t) \leq -\alpha \bar{\mathcal{V}}_i(t) \tag{40}$$

*Integrating the above inequality for any $t \in [t_k, t_{k+1})$ gives that*

$$\bar{\mathcal{V}}_i\left(x\left(t_{k+1}\right)\right) \leq \bar{\mathcal{V}}_i\left(x\left(t_k\right)\right) e^{-\alpha\left(t_{k+1} - t_k\right)} \tag{41}$$

*Thus, over the interval $[t_0, t]$ and according to* (36)*,* (41) *and $\mathcal{N}_\sigma(t_0, t) \leq \mathcal{N}_0 + (t - t_0)/\tau_a$, it follows that*

$$\begin{aligned} \bar{\mathcal{V}}_i(x(t)) &\leq e^{-\alpha(t - t_k)} \bar{\mathcal{V}}_i\left(x\left(t_k\right)\right) \\ &\leq \mu e^{-\alpha(t - t_k)} \bar{\mathcal{V}}_{i-1}\left(x\left(t_k^-\right)\right) \\ &\quad \vdots \\ &\leq \mu^{\mathcal{N}_\sigma(t_0, t)} e^{-\alpha(t - t_0)} \bar{\mathcal{V}}_0\left(x\left(t_0\right)\right) \\ &\leq \mu^{\mathcal{N}_0} e^{-(\alpha - \ln(\mu)/\tau_a)(t - t_0)} \bar{\mathcal{V}}_0\left(x\left(t_0\right)\right) \end{aligned} \tag{42}$$

*Therefore, if the average dwell time (10) is satisfied, the state* $x(t) \to 0$ *when* $t \to \infty$*, which means that the system (33) is asymptotically stable.*
*Now, we set* $\tilde{\mathcal{J}}_2(t) = -x^T(t)\mathcal{W}x(t) + \gamma_1^2\rho^T(t)\rho(t)$*, for any* $\rho(t) \in l_2[0,\infty)$ *and with* $x_0 = 0$*. It is noted that*

$$\dot{\mathcal{V}}_i(t) + \alpha\mathcal{V}_i(t) - \tilde{\mathcal{J}}_2(t) \leq \quad \eta^T(t)\Theta_i\eta(t) \tag{43}$$

*where*

$$\Theta_i = \begin{bmatrix} \mathcal{A}_{ci}^T\bar{\mathcal{P}}_i + \bar{\mathcal{P}}_i\mathcal{A}_{ci} + \alpha\bar{\mathcal{P}}_i + \mathcal{W} & \bar{\mathcal{P}}_i\mathcal{E}_i \\ * & -\gamma_2^2\mathbb{I} \end{bmatrix}, \eta(t) = \begin{bmatrix} x(t) \\ \rho(t) \end{bmatrix}$$

.
*Pre- and Post-multiplying* $\Theta_i$ *by* $diag\{\bar{\mathcal{P}}_i^{-1}, \mathbb{I}\}$ *and its transpose, the following inequality holds:*

$$\begin{bmatrix} \bar{\mathcal{P}}_i^{-1}\mathcal{A}_{ci}^T + \mathcal{A}_{ci}\bar{\mathcal{P}}_i^{-1} + \bar{\mathcal{P}}_i^{-1}\mathcal{W}\bar{\mathcal{P}}_i^{-1} + \alpha\bar{\mathcal{P}}_i^{-1} & \mathcal{E}_i \\ * & -\gamma_2^2\mathbb{I} \end{bmatrix}$$

*Denote*

$$\mathcal{X}_i = \bar{\mathcal{P}}_i^{-1} \tag{44}$$

*Using schur complement, the inequality (35) is obtained. If (35) is satisfied, one get:*

$$\dot{\bar{\mathcal{V}}}_i(t) + \alpha\bar{\mathcal{V}}_i(t) - \tilde{\mathcal{J}}_2(t) \leq \quad 0 \tag{45}$$

*By Lemma 2.3, we have*

$$\bar{\mathcal{V}}_i(t) \leq e^{-\alpha(t-t_k)}\bar{\mathcal{V}}_i(t_k) + \int_{t_k}^{t} e^{-\alpha(t-s)}\tilde{\mathcal{J}}_2(s)\mathrm{d}s \tag{46}$$

*Based on the same reasoning given in the proof of Corollary 3.1, one can obtain*

$$\int_{t_0}^{t} e^{-\alpha(t-s)-\alpha s}x^T(s)\mathcal{W}x(s)\mathrm{d}s \leq \gamma_2^2\int_{t_0}^{t} e^{-\alpha(t-s)}\rho^T(s)\rho(s)\mathrm{d}s \tag{47}$$

*For* $t_0 = 0$ *and* $t \to \infty$*, Inequality* (34) *is obtained. To finalize the proof, we multiply Inequality* $(\mathcal{A}_i+\mathcal{B}_i\mathcal{G}_i)$ *by* $\bar{\mathcal{P}}_i^{-1}$ *(on the right), this can be rewritten as* $(\mathcal{A}_i\bar{\mathcal{P}}i^{-1}+\mathcal{B}_i\mathcal{G}_i\bar{\mathcal{P}}_i^{-1})$*, where* $X_i = \bar{\mathcal{P}}_i^{-1}$*,* $Y_i = \mathcal{G}_i\bar{\mathcal{P}}_i^{-1}$*, then (36) is derived. The proof is completed.*

## 4 Data Communication Network Example

In this section, we examine the issue of Fault-Tolerant Control (FTC) applied to a data communication network composed of three nodes [14]. A typical scenario is considered, in which a large data packet is transmitted from control center A to communication terminal B via a switched network. This network alternates between two operational modes as shown in Figure 1, i.e. idle mode and busy mode.

When a network node carries more data than it can handle, congestion may occur, leading to the loss of data packets. Conversely, transmitting too little data results in inefficient use of network resources. To tackle this issue, a switched positive system model of the form $\dot{x}(t) = \mathcal{A}_{\sigma(t)}x(t)$ has been proposed, where $x(t) = [x_1(t), x_2(t), x_3(t)]^T$ denotes the amount of data transmitted by each node per unit time. The work presented in [19] extended this model by introducing a control input $u(t)$, designed to discard data packets when congestion occurs or to send new packets when the transmitted data is below the network's capacity.

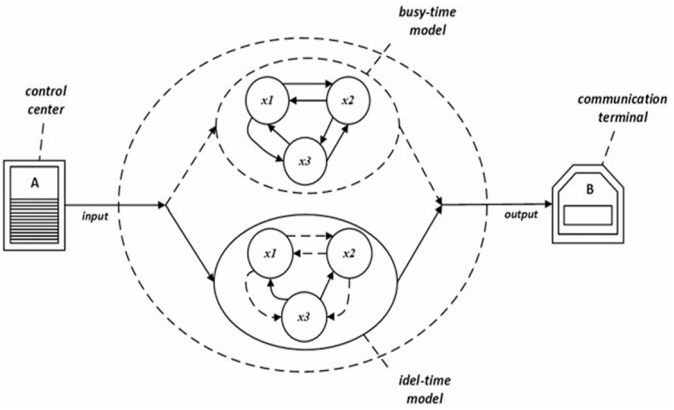


Figure 1: Data communication network composing of three nodes.

To enhance the model's realism and evaluate the robustness of the FTC design, we introduce in this paper an an actuator additive fault, denoted by $f(t)$, which is used to represent a network active attack scenario. In this case, a malicious entity forces the control center to send an unbearable amount of data packets, exceeding the physical capacity of the network, which can lead to severe congestion and packet loss. This fault is additive in nature and directly affects the stability of system. The configuration described is appropriately represented by system (1), where matrices $A_i$, $B_i$ and $C_i$ are given as follows [14]:

$$\begin{aligned}
\mathcal{A}_1 &= \begin{bmatrix} -2 & 1.4 & 0.6 \\ 0.9 & -1 & 0.5 \\ 0.1 & 0.4 & -0.7 \end{bmatrix}, \mathcal{A}_2 = \begin{bmatrix} -2 & 0.2 & 2 \\ 0.35 & -1 & 0.2 \\ 0.1 & 2 & -3.7 \end{bmatrix}, \mathcal{B}_1 = \begin{bmatrix} 0.2 \\ 0.4 \\ 0.15 \end{bmatrix}, \\
\mathcal{B}_2 &= \begin{bmatrix} 0.3 \\ 0.1 \\ 0.2 \end{bmatrix}, C_1 = \begin{bmatrix} 0.01 & 0.05 & 0.04 \\ 0.01 & 0.05 & 0.02 \\ 0.25 & 0.03 & 0.06 \end{bmatrix}, C_2 = \begin{bmatrix} 0.02 & 0.01 & 0.03 \\ 0.01 & 0.02 & 0.03 \\ 0.05 & 0.04 & 0.01 \end{bmatrix},
\end{aligned}$$

Moreover, The disturbance considered in this paper represents random and unpredictable variations in network traffic, such as sudden spikes in data flow, background traffic from other systems, or environmental interference. The disturbance matrices are given by:

$$\mathcal{E}_1 = \begin{bmatrix} 0.2 \\ 0.12 \\ 0.3 \end{bmatrix}, \mathcal{E}_2 = \begin{bmatrix} 0.1 \\ 0.14 \\ 0.4 \end{bmatrix},$$

Taking $\gamma_1 = 0.5$, $\gamma_2 = 0.4$, $\epsilon_1 = 0.3$, $\epsilon_2 = 0.2$, $\alpha = 0.8$ and $\mu = 1.6$, the parameters of the SPIO and fault tolerant controller are given as follows:

$$\begin{aligned}
\mathcal{G}_1 &= \begin{bmatrix} -1.25 & 13.5 & 11.91 \end{bmatrix}, \mathcal{G}_2 = \begin{bmatrix} 7.90 & 14.5 & 14.33 \end{bmatrix}, \\
\mathcal{K}_{11} &= \begin{bmatrix} -0.140 & -0.556 & 0.231 \end{bmatrix}, \mathcal{K}_{12} = \begin{bmatrix} 0.181 & -0.5165 & 0.733 \end{bmatrix}, \\
\mathcal{K}_{21} &= \begin{bmatrix} -43.047 & 32.782 & 93.238 \end{bmatrix}, \mathcal{K}_{22} = \begin{bmatrix} 41.77 & -20.61 & 88.602 \end{bmatrix}, \\
\mathcal{L}_1 &= \begin{bmatrix} 502.678 \\ 12.01662 \\ 149.68 \end{bmatrix}, \mathcal{L}_2 = \begin{bmatrix} 837.259 \\ 860.980 \\ 809.968 \end{bmatrix}, \Lambda_1 = 465.56, \Lambda_2 = 225.67,
\end{aligned}$$

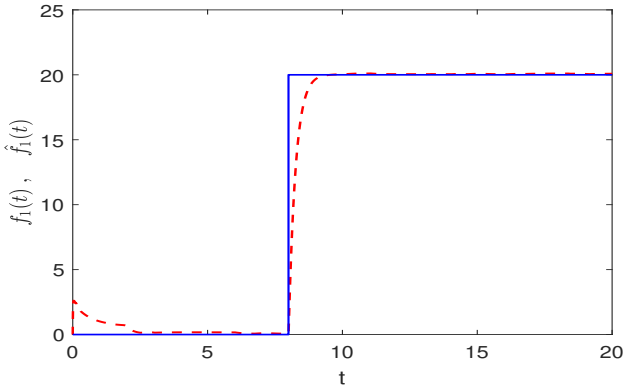


Figure 2: fault $f_1(t)$ (solid line) and its estimate (dotted line).

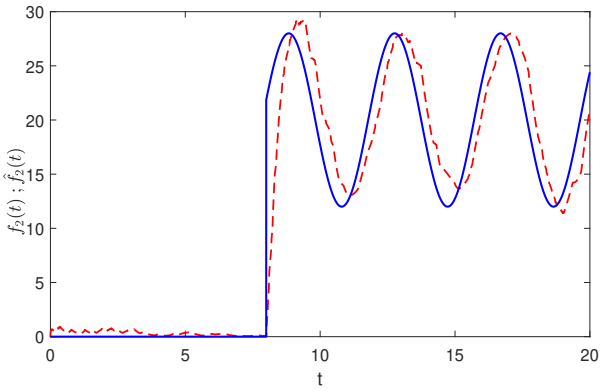


Figure 3: fault $f_2(t)$ (solid line) and its estimate (dotted line).

In the simulation, the disturbance is modeled as a random signal with a magnitude of 0.8. If the fault occur, we can present it as follows:

$$f_1(t) = \begin{cases} 0 & 0 \le t \le 8 \\ 20 & 8 < t \le 20 \end{cases}$$

$$f_2(t) = \begin{cases} 0 & 0 \le t \le 8 \\ 20 + 8sin(1.6t) & 8 < t \le 20 \end{cases}$$

Figures 2 and 3 demonstrate that both constant and time-varying faults are accurately and rapidly estimated, even in the presence of unknown inputs. In Figures 4-6, the asymptotic stability of the closed-loop system is preserved despite actuator faults. Moreover, the system states and the observer remain non-negative, confirming that the required conditions are satisfied.

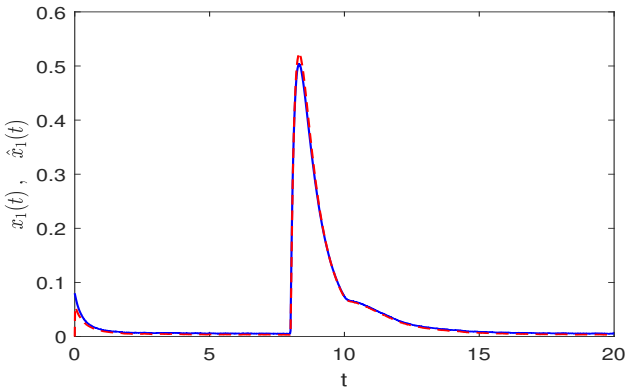


Figure 4: state $x_1(t)$ (solid line) and its estimate (dotted line).

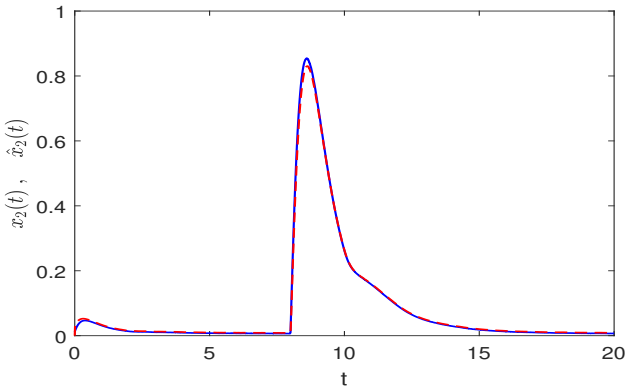


Figure 5: state $x_2(t)$ (solid line) and its estimate (dotted line).

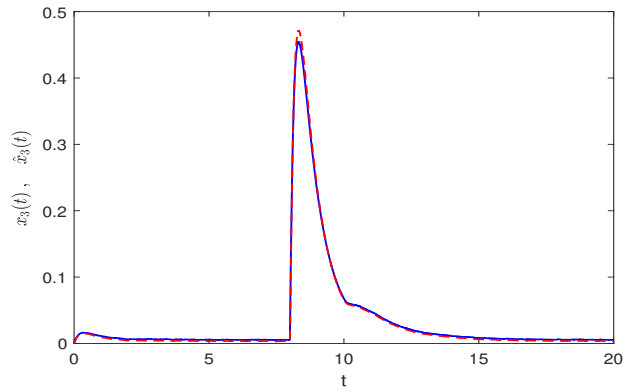


Figure 6: state $x_3(t)$ (solid line) and its estimate (dotted line).

## 5 CONCLUSION

This paper is devoted to treat the issue of robust FTC with Unknown input and positivity conditions. Using multiple Lyapunov function and the ADT constraints. Hence, the designed observers gains satisfy the stability and positivity of the augmented system while the robustness conditions are respected. Finally, a practical example is given to prove the effectiveness of the developed results. As perspectives of this paper, we believe that an extension to FTC based on neural network observer can be done using the same technique developed in [20].

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