Continuous Reformulation of Planck's Law.

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Abstract. In this paper, we present a continuous reformulation of Planck's quantization law within a continuous and geometric thermodynamic framework. The discrete spectrum $E_n = nh\nu$ that underlies the traditional Planck distribution is generalized into a functional form $E_n = E_0 f(n)$, where $f(n)$ defines the intrinsic geometry of the accessible energy states. This continuous formalism embeds Boltzmann, Gibbs, and Planck statistics across arbitrary spectral geometries, where quantization emerges as a geometric reinterpretation of energy itself. For the exponential case $f(n) = e^{\lambda n}$, where λ encodes an intrinsic geometric scale, the Planck distribution is recovered in the flattening limit $\lambda \to 0$ with $E_0\lambda$ held fixed, so that $E_n \approx nh\nu$ up to a constant offset. Non-zero curvature then describes self-similar and non-thermal spectra. More generally, any smooth spectral law that becomes locally linear recovers the standard Planck regime as its zero-curvature limit. Within this representation, the canonical partition function admits a Mellin-type *continuum approximation* for monotonic and differentiable spectral laws, and the blackbody law emerges as a limiting case of a more general, self-similar spectral thermodynamics. The formulation provides a coherent geometric interpretation of Boltzmann statistics, Gibbs' partition function, and Planck's quantization, extending them to systems with non-uniform confinement or curvature. Quantization thus appears not as a discrete postulate, but as a geometric property of the energy manifold.

1 Introduction

The birth of quantum theory was marked by Planck's discrete quantization [1, 2], together with the associated canonical averaging that led to the blackbody radiation law:

$$E_n = nh\nu, \tag{1}$$

where each oscillator of frequency ν in a thermal cavity can only hold integer multiples of the elementary quantum $h\nu$. Thermal equilibrium is then entirely characterized by the temperature T, and the material details of the cavity walls play no dynamical role; the spectrum is assumed universal and purely thermal.

However, many radiative systems of physical interest are *not* purely thermal or universal. Plasmonic cavities, nanoresonators, driven emitters, near-field thermal devices, or structured photonic media display spectra that are neither perfectly Planckian nor strictly discrete-line atomic spectra. Their spectra combine continuous emission envelopes with scale-dependent resonant peaks. Such systems therefore fall outside the purely discrete assumption of Eq. (1).

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In recent work, we introduced a generalized statistical framework in which the canonical Boltzmann–Gibbs weight is preserved, but the spectrum of accessible energies is no longer assumed to be additive or evenly spaced [5]. This framework was subsequently extended into a full thermodynamic theory of spectral energy, where the energy levels follow an exponential law and the natural analytic tool is the Mellin transform rather than the standard Laplace transform [6]. Within this approach, the resulting formalism describes a *model-based self-similar spectral thermodynamics*, in which self-similarity emerges from the chosen spectral geometry rather than from direct experimental observation.

In the present paper, we reformulate Planck quantization and extract a geometric interpretation of the radiation law. Planck's law can be understood as the limiting case of a broader, geometrically organized energy spectrum. Accordingly, instead of postulating discrete additive quanta, we introduce a continuous energy law that varies across scales. In this picture, quantization is not imposed by hand as a linear ladder but emerges, *within the present model*, as a manifestation of curvature within an abstract energy manifold.

2 From discrete quanta to continuous spectra

In Planck's original derivation, the allowed energy values of an electromagnetic mode of frequency ν are discrete and uniformly spaced, following the linear law (1). The corresponding canonical partition function reads

$$Z_{\text{Planck}} = \sum_{n=0}^{\infty} \exp\left(-\frac{nh\nu}{k_B T}\right) = \frac{1}{1-\exp(-h\nu/k_B T)}. \tag{2}$$

From $\ln Z_{\text{Planck}}$, one obtains the mean energy $\langle E \rangle$ and recovers the standard Planck distribution for blackbody radiation.

The central assumption behind (2) is *additivity*: each energy level differs from the previous one by the same increment $h\nu$. We now relax this constraint.

Let the spectrum be defined more generally by

$$E_n = E_0 f(n), \tag{3}$$

where $f(n)$ need not be linear in n, but must ensure the convergence of the canonical partition function $Z_f(s)$. The corresponding partition function becomes

$$Z_f(s) = \sum_{n=0}^{\infty} \exp\left[-\frac{sE_0 f(n)}{k_B T}\right], \tag{4}$$

where s is a dimensionless scaling parameter (e.g., mode degeneracy, coupling weight, or geometric multiplicity). The canonical Boltzmann weighting is preserved, while the spectral law is generalized. Equation (4) reduces to (2) when $f(n) = n$ and $s = 1$.

The expressions $E_n = E_0 f(n)$ and $Z_f(s)$ go beyond a simple reparametrization of existing models; they redefine the structure of statistical mechanics by treating the energy spectrum as a geometric property of the thermodynamic manifold. Moving from fixed linear spectra to continuous laws that vary across scales extends Boltzmann and Planck formulations into a coherent geometric framework.

A particularly relevant case, introduced in [5–7], is the *exponential spectrum*

$$f(n) = e^{\lambda n} \quad \Longrightarrow \quad E_n = E_0 e^{\lambda n}, \tag{5}$$

where λ is a (dimensionless) scale parameter. Instead of a uniform spacing $E_{n+1} - E_n = h\nu$, the levels form a geometric progression with the ratio

$$\frac{E_{n+1}}{E_n} = e^{\lambda}. \tag{6}$$

Equivalently, the spacing grows multiplicatively,

$$E_{n+1} - E_n = (e^{\lambda} - 1)\, E_n, \tag{7}$$

so the spectrum is self-similar across scales rather than evenly spaced. This multiplicative hierarchy signals an intrinsic curvature within the space of accessible energies.

In the flattening limit $\lambda \to 0$ with $E_0\lambda$ held fixed, one obtains

$$E_n = E_0(1 + \lambda n) \;\longrightarrow\; E_n \approx nh\nu \;+\; \text{const.}, \tag{8}$$

so that the exponential spectrum continuously reduces to the linear Planck quantization up to a constant offset. Thus, Planck's law appears as the zero-curvature limit of a broader, scale-dependent thermodynamic geometry.

In the flat or weak-curvature limit, any admissible spectral function $f(n)$ that is differentiable and approximately linear in n recovers the classical Planck form. Indeed, expanding $f(n)$ to first order gives

$$f(n) \simeq f(0) + f'(0)\, n, \tag{9}$$

so that

$$E_n = E_0 f(n) \simeq E_0 f(0) + E_0 f'(0)\, n. \tag{10}$$

By identifying $E_0 f'(0) = h\nu$ and neglecting the constant offset $E_0 f(0)$, one obtains the linear quantization rule

$$E_n \simeq nh\nu, \tag{11}$$

which defines the Planck regime. Therefore, the discrete linear law arises as the *zero-curvature limit* of any continuous spectral geometry admitting a first-order linear expansion. The exponential case $f(n) = e^{\lambda n}$ illustrates this principle explicitly [7], with λ acting as a curvature parameter whose vanishing restores the linear spectrum.

3 Continuous partition and Mellin structure

In the previous section, we introduced the general spectral form (3) and the corresponding canonical sum (4). Different choices of $f(n)$ define distinct thermodynamic geometries. Among them, the exponential law

$$f(n) = e^{\lambda n} \quad \Longrightarrow \quad E_n = E_0 e^{\lambda n},$$

plays a special role.

Among the possible spectral laws, the exponential form $f(n) = e^{\lambda n}$ plays a distinguished role: it is the simplest functional form for which the canonical partition sum *admits a Mellin-type continuum approximation* via the change of variables $x = e^{\lambda n}$, where the weighting becomes multiplicative rather than additive. Other monotonic or power-law forms of $f(n)$ may also lead to Mellin-like structures under suitable transformations, but the exponential case remains the simplest and most analytically tractable realization. This property enables a direct analytic continuation of the discrete sum into a continuous integral with a logarithmic measure, making explicit the geometric nature of the energy scaling.

Inserting $f(n) = e^{\lambda n}$ into $Z_f(s)$ yields

$$Z_\lambda(s) = \sum_{n=0}^{\infty} \exp\left[-\frac{sE_0 e^{\lambda n}}{k_B T}\right], \tag{12}$$

which defines a "double-exponential" partition function characteristic of self-similar spectra. This is no longer a geometric series: the Boltzmann factor is applied to an exponentially growing spectrum. As shown in [6] and following the Mellin–zeta framework introduced by Riemann [4], Eq. (12) can be treated through a Mellin-type analysis.

Let $x = e^{\lambda n}$ so that $n = \frac{1}{\lambda} \ln x$ and $\mathrm{d}n = \frac{1}{\lambda} \frac{\mathrm{d}x}{x}$. Applying the *Euler–Maclaurin formula* to replace the discrete sum by its continuum approximation gives

$$Z_\lambda(s) \approx \frac{1}{\lambda} \int_1^{\infty} x^{-1} \exp\left[-\frac{sE_0 x}{k_B T}\right] \mathrm{d}x. \tag{13}$$

This integral has the structure of a Mellin transform, i.e. an integral of the form $\int x^{u-1} F(x)\, \mathrm{d}x$. In this sense, Eq. (12) admits a Mellin-type representation rather than the usual Laplace structure of standard canonical ensembles. The thermodynamics thus ceases to be purely additive in energy and becomes multiplicative across scales.

From $Z_\lambda(s)$, one can still define the mean energy

$$\langle E \rangle_\lambda = \frac{1}{Z_\lambda(s)} \sum_{n=0}^{\infty} E_0 e^{\lambda n} \exp\left[-\frac{sE_0 e^{\lambda n}}{k_B T}\right], \tag{14}$$

and the associated entropy in canonical form,

$$S_\lambda = k_B \left(\ln Z_\lambda(s) + \frac{\langle E \rangle_\lambda}{k_B T} \right). \tag{15}$$

Equations (12)–(15) generalize the familiar Gibbs–Boltzmann relations to a non-additive, scale-structured spectrum. In the limit $\lambda \to 0$, the multiplicative increment $e^\lambda \to 1 + \lambda$ and the exponential ladder (5) flattens into a linear one. Then $Z_\lambda(s)$ continuously reduces to Z_{Planck} in (2), and the standard Planck law is recovered. Planck's law therefore appears as the *zero-curvature limit* of a broader, scale-dependent thermodynamic structure.

4 Geometric interpretation of the spectrum

The exponential law (5) admits a natural geometric interpretation. If consecutive energy levels differ by a constant *ratio* rather than a constant difference, the spectrum is organized into a hierarchy of scales. Such hierarchies are naturally associated, by analogy, with spaces of negative curvature [3], where distances grow multiplicatively rather than additively.

A convenient analogy is provided by the Poincaré disk, which offers a simple model of multiplicative scaling in a negatively curved space. In this geometry, the hyperbolic distance between two radial points z_1 and z_2 is given by

$$d_H(z_1, z_2) = \operatorname{arcosh}\left(1 + \frac{2|z_1 - z_2|^2}{(1 - |z_1|^2)(1 - |z_2|^2)}\right). \tag{16}$$

Moving radially outward corresponds to exploring successively higher "scales" rather than linear coordinates. This is directly analogous to the transition $E_n \to E_{n+1}$ in (5), where each step rescales the energy by a fixed factor e^λ.

In this analogy, the parameter λ acts as a curvature-like quantity controlling the deviation from the flat spectrum. Larger values of λ correspond to stronger geometric deformation of the energy spacing, whereas the limit $\lambda \to 0$ represents a flattening of the spectrum toward linear quantization. By analogy with hyperbolic geometry, this behavior can be summarized as an effective relation

$$\kappa_{\text{eff}} \sim -\lambda^2, \tag{17}$$

where κ_{eff} denotes an *effective* curvature parameter with no claim of direct physical measurement.

When $\lambda \neq 0$, the energy manifold thus behaves, by analogy with a hyperbolic space, as if it possessed a finite negative curvature: the accessible states populate a multiplicative, scale-structured ladder. In the limit $\lambda \to 0$, this curvature tends to zero, the ladder becomes linear, and one recovers the evenly spaced spectrum of Planck quantization.

This geometric reading reinterprets quantization, within the present model, as a manifestation of curvature in an abstract energy space: discreteness is not a primitive postulate, but a special geometric regime corresponding to the flat limit of the thermodynamic manifold.

5 Discussion and physical implications

Our continuous spectral formulation (5)–(17) provides a coherent framework for describing radiative systems beyond the thermal equilibrium assumption, including non-thermal and geometrically confined emitters [7].

First, it naturally accommodates *non-thermal* or *quasi-thermal* emitters. Classical blackbody radiation assumes a cavity in global equilibrium and a spectrum that depends only on the temperature T. In contrast, real systems such as nanoresonators, plasmonic gaps, and near-field thermal emitters exhibit spatial confinement and mode-dependent enhancement, leading to noticeably non-Planckian or programmable emissive properties [8–10]. In such contexts, energy exchange is not governed by a single global quantum $h\nu$, but by multiple local modes or geometric scales that shape the spectral response. Within the present framework, the curvature parameter λ encodes this confinement or spectral hierarchy of electromagnetic modes.

Second, the Mellin-type partition function (12)–(13) suggests that statistical mechanics in structured radiative media is not purely additive but organized by multiplicative energy scaling. This defines a *thermodynamics of scales*, where the relevant degrees of freedom are distributed across hierarchical energy manifolds rather than evenly spaced levels. The same reasoning could extend to fractal optical media, hierarchical resonator arrays, or multiscale plasmas.

Finally, in the $\lambda \to 0$ limit, all these deformations continuously reduce to the standard Planck law. Planck's spectrum is therefore not contradicted but embedded as the flat, additive limit of a broader, curved, and multiplicative thermodynamic geometry.

6 Conclusion

We have shown that Planck's discrete quantization can be reformulated as the limiting case of a continuous, geometrically organized energy spectrum. Replacing the linear law $E_n = nh\nu$ by a general functional form $E_n = E_0 f(n)$ allows the energy manifold itself to acquire a geometry. In this framework, the canonical partition function becomes a spectral object whose analytic structure depends on the intrinsic form of $f(n)$. For exponential spectra, the partition assumes a Mellin-type form, introducing a scale-dependent entropy S_λ and a curvature $\kappa_{\text{eff}} \sim -\lambda^2$ that characterizes the underlying hyperbolic geometry. More generally,

any smooth $f(n)$ that admits a local linear expansion recovers the standard Planck regime in the zero-curvature limit.

Quantization thus appears not as a fundamental discreteness, but as a geometric manifestation of curvature in the energy manifold. The classical blackbody spectrum emerges as the flat limit of a broader, continuous thermodynamic geometry.

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