An improved control approach for a class of continuous-time switched system with time-varying delay

EL MAACH Amina[1,*], *EL AZZOUZI* Salma[1], *EL AAZZAOUI* Omar[1], *BOUKILI* Bensalem[1], and *CHARQI* Mohammed[1,**]

[1]LESSI Laboratory, Faculty of Science Dhar El Mehraz, Sidi Mohamed Ben Abdellah University, Fez, Morocco.

Abstract. In this paper, an improved method is developed for designing controllers for linear switched systems with time-varying delays, focusing on the Hurwitz-convex combination framework. For this, by using a Lyapunov–Krasovskii functional, the stability analysis is carried out, and the obtained conditions are expressed in terms of Linear Matrix Inequalities (LMIs). By solving these LMIs, one can determine an upper bound on the delay that ensures the stability of the considered system. Compared to some existing results in the literature, the proposed approach yields less conservative results and allows for significantly larger admissible delay bounds. Finally, some numerical examples are provided to demonstrate the effectiveness and to improve the performance of the proposed design scheme.

Index Terms: Switched Systems, Linear Matrix Inequalities (LMIs), Lyapunov-Krasovskii functions, Time-delay, Stability, Control design.

1 Introduction

Switched systems (SSs) represent an important class of hybrid dynamical systems that arise in various real-world applications, such as networked control systems, power electronics, traffic control, communication-based industrial processes [14], [13] and control of mechanical systems [10], [15]. These systems consist of a finite number of distinct active subsystems and a switching rule that dictates the active subsystem at any given time [1], [4], [8]. The presence of time-varying delays; caused by many factors such as actuator delays, sensor transmission lag, or network congestion; can generates significant complexity into both the stability analysis and control design of switched systems with time-delay. Such delays, if not properly handled, can degrade performance and potentially destabilize the system, see for example [6] and the references therein.

On the other hand, switching signal plays a crucial role in the dynamics of switched systems. however, different switching strategies have been proposed in the literature, including arbitrary switching, dwell-time switching, average dwell-time switching, mode-dependent switching, and state-dependent switching [8], [11]. Since arbitrary switching allows unrestricted switching behavior, it presents substantial challenges for stability verification and control design, see, eg. [2], [3]. In contrast, dwell-time switching requires a minimum time duration between two consecutive switches, which helps in guaranteeing stability [12]. State-dependent switching uses the state of the system to determine the next active mode and is often applied to improve performance or avoid unsafe behaviors, see for example [23], [24]. Each type of switching influences the system's stability and affects the complexity of controller design.

*e-mail: amina.elmaach@usmba.ac.ma
**e-mail: mohammed.charqi@usmba.ac.ma

Besides, time-delay phenomena, can have a profound impact on system behavior. Delays introduce memory into the system, meaning that past states affect future dynamics. This can lead to oscillations, instability, or degraded transient response [7], [24]. They are encountered in many several engineering applications such as chemical processes, nuclear reactor, aerospace systems, and neural network, and so on [5] and [7]. As a result, stability analysis becomes more challenging, which allowed the development of several approaches that analyse the stability of delayed systems, such stability is done in two methods : delay-dependent or delay-independent [9] and [25], including the use of Lyapunov–Krasovskii functionals, which allow for the incorporation of time-delay information in the analysis. It is also well known that the stability analysis of switched systems with time-varying delays presents significant challenges due to the combined effects of switching logic and delay-induced memory. In such systems, the presence of variable delays introduces complex dependencies on the past states, which may result in oscillatory behaviors, performance degradation, or even instability. To address these difficulties, Lyapunov-based methods, particularly those relying on Lyapunov–Krasovskii functionals [2], [3] and [16], have proven to be powerful and widely adopted.

It should be noted that control of switched systems with time-varying delay is an active area of research due to the complex interplay between discrete switching logic and continuous dynamics with delays. Designing controllers that ensure stability and performance under uncertain or time-varying delays requires robust and often adaptive strategies. Common methods include state-feedback control, output feedback, and observer-based control [17], [18], [19] and other recent references. Advanced techniques such as mode-dependent controllers, delay compensation, and predictive control are also employed to enhance robustness [20]. Ultimately, ensuring stability and satisfactory performance in the presence of both switching and delays remains a fundamental challenge in modern control theory.

The integration of linear matrix inequalities (LMIs) with Lyapunov–Krasovskii functionals facilitates the formulation of computationally tractable and verifiable conditions. Recent contributions have aimed at reducing conservatism by introducing slack variables, mode-dependent Lyapunov structures, and switching-aware delay bounds. It is well established that the inclusion of time delays significantly increases the complexity of switched systems compared to their delay-free counterparts. In recent years, considerable attention has been devoted to the stability analysis of linear switched systems with delays. For instance, the authors of [21] and [22] have proposed a method for stability analysis and robust model reference tracking control of nonlinear switched time-delay systems. Moreover, Sun et al. in [27] developed an LMI-based approach that is less conservative than the one presented by the authors of [28]. Specifically, the method in [27] is applicable to linear switched systems with time-varying delays, whereas the method in [28] is restricted to systems with constant delays. These last two methods have been improved by Zabihi et al. [24], by developing a new method for designing controller for linear switching systems with time-varying delay. Nonetheless, the delay bounds obtained from the criteria proposed in [24], [28] and [27], while ensuring stability, remain very restrictive. This limitation constitutes the main motivation of the present work. Accordingly, this paper proposes a controller design for linear switched time-delay systems that not only guarantees stability but also achieves satisfactory performance in the presence of delays with significantly larger upper bounds compared with existing approaches, the method proposed in this paper is less conservative and effectively addresses the limitation imposed by small delay bounds in the stability analysis of linear switched systems.

The stability analysis is carried out using a Lyapunov–Krasovskii functional, introduced in Section 3, with the corresponding conditions expressed in terms of linear matrix inequal-

ities (LMIs). The admissible delay bound is then determined by solving these LMIs, and using ready-to-use software: LMI Lab/MATLAB and Simulink/MATLAB.

Notation 1 . $\mathbb{R}^n$ *denotes the n-dimensional Euclidean space and* $\mathbb{R}^{n\times m}$ *is a real* $n\times m$ *matrix.* $X > 0$ (≥ 0) *means that matrix X is positive definite (positive semidefinite) and* $X < 0$ (≤ 0) *indicates that matrix X is negative definite (negative semidefinite).* I_p *denotes the identity matrix of* $p\times p$ *dimension.* $\lambda_{max}(X)$ *denotes maximum eigenvalue of matrix X and* $\lambda_{min}(X)$ *represents minimum eigenvalue of matrix X. The symmetric terms in a symmetric matrix is denoted by '*$*$*'. Superscript* $'T'$ *stands for matrix transposition.* $Sym\{X\}$ *denotes* $X + X^T$ *for any square real matrix X.*

2 Problem Statement and Preliminaries

In this study, it is assumed that all state variables are available, and the following class of switching systems with time-varying delays is investigated :

$$\begin{cases} \dot{x}(t) = A_i x(t) + A_{hi} x(t - h(t)) + u_i(t), i \in N = \{1, 2, ..., l\}, \\ x(\theta) \;=\; \phi(\theta), \quad \theta \in [-\tau,\ 0], \end{cases} \tag{1}$$

where, $x(t) \in \mathbb{R}^n$ is the state vector, $\phi(k) \in \mathbb{R}^n$ is a given initial condition sequence and $u_i(t) \in \mathbb{R}^n$ denote the controllers of subsystems for $i \in N$. A_i, $A_{hi} \in \mathbb{R}^{n\times n}$ are real known matrices. $h(t)$ represents time-varying delay that is differentiable and satisfying the following conditions $0 \leq h(t) \leq \tau$ with $\tau > 0$ is the upper bound of delay. The derivative of time-varying delay is $\dot{h}(t)$ which satisfies $\dot{h}(t) \leq h < 1$ for known constant scalar h.

In accordance with the conventional notation outlined in [1], the sequence of switching is described as follows:

$$\sum = \{x_0, (i_0, t_0), (i_1, t_1), ..., (i_q, t_q), .../i_q \in N, q = 0, 1, 2, ..\} \tag{2}$$

$$\sigma(t) = i,\ if\ x(t) \in \Lambda_i,\ i \in N = \{1, 2, ..., l\}. \tag{3}$$

$$\sigma(t) : \mathbb{R}^+ \rightarrow N = \{1, 2, ..., l\}.$$

Where x_0 is initial state and t_0 is initial time. In the case where $t \in [t_q, t_{q+1}]$, $x(t)$ belongs to Λ_{i_q}, which leads to the activation of the subsystem i_q^{th}. $\sigma(t)$ which can be either time-dependent or state-dependent at any given moment, it designates the active subsystem. This paper, takes into consideration the state-dependent case for switching signal, which is specified in equation (3).

2.1 Hurwitz convex combination

Following the strategies outlined in the references [26], [27] and [28], We assume the existence of a Hurwitz convex combination F of the matrices $A_i + A_{hi}$ which guarantees the following conditions :

$$F = \sum_{i=1}^{n} \alpha_i (A_i + A_{hi}) \tag{4}$$

where, $0 < \alpha_i < 1$ and $\sum_{i=1}^{l} \alpha_i = 1\ for\ all\ \ i \in N = \{1, 2, ..., l\}$.

Owing to the Hurwitz nature of F, a positive definite matrix P exists that satisfies the following equation for a chosen positive definite matrix H :

$$F^T P + PF = -H \tag{5}$$

2.2 Switching regions

For given matrices P and H, the following equation defines the switching region :

$$\Lambda_i = \{x^T \in \mathbb{R}^n / x^T[(A_i + A_{hi})^T P + P(A_i + A_{hi})]x \leq -x^T Hx\}. \tag{6}$$

From the condition $H > 0$, and by substituting equation (4) into the previous equation (5), it can be proved that the inequality $\alpha_i x^T[(A_i + A_{hi})^T P + P(A_i + A_{hi})]x < 0$ holds for at least one subsystem. In [28], this result is further reinforced through a proof by contradiction, which demonstrates that the union of the switching regions spans the entire space $\mathbb{R}$, i.e., $\bigcup_{i=1}^{l} \Lambda_i = \mathbb{R}^n$.

According to the switching region designed in equation (6), fast switching events should occur, which causes a significant problem. Specifically, a great number of transitions between subsystems within a short time interval can induce sliding motions and chattering phenomena along the state trajectories near the boundaries separating the switching regions. To overcome this unexpected behavior, it is advisable to design overlapping switching regions, which effectively slow down the switching process at the boundaries. This overlap helps reduce the probability of rapid switching and thus mitigates the phenomenon of state trajectories slipping near region boundaries.

In order to establish an overlapping region, the switching regions are redefined as follows:

$$\tilde{\Lambda}_i = \{x^T \in \mathbb{R}^n / x^T[(A_i + A_{hi})^T P + P(A_i + A_{hi})]x \leq -\frac{1}{\varsigma} x^T Hx\} \tag{7}$$

In the previous inequality (7), ς is taken as $\varsigma > 1$. Then, multiplying $\frac{1}{\varsigma}$ by $-x^T Hx$ implies an enlargement of the result $-\frac{1}{\varsigma} x^T Hx$. This means that the value approaches zero, resulting in a greater number of state vectors satisfying condition (7), which implies that the switching regions are effectively enlarged, leading to the formation of overlapping areas. In fact, $\Lambda_i \subset \tilde{\Lambda}_i$, and since $\bigcup_{i=1}^{l} \Lambda_i = \mathbb{R}^n$, on can gets $\bigcup_{i=1}^{l} \tilde{\Lambda}_i = \mathbb{R}^n$. In other words, given that $\varsigma > 1$ and $x^T Hx > 0 \ \forall x \neq 0$, therefore :

$$-\frac{1}{\varsigma} x^T Hx \geq -x^T Hx. \tag{8}$$

Using both equations (6) and (8), we obtain the following inequalities :

$$x^T[(A_i + A_{hi})^T P + P(A_i + A_{hi})]x \leq -\tfrac{1}{\varsigma} x^T Hx.$$

For the reason that $\bigcup_{i=1}^{l} \Lambda_i = \mathbb{R}^n$, as a result $\bigcup_{i=1}^{l} \tilde{\Lambda}_i = \mathbb{R}^n$.

Consequently, the switching region in (7) is verified by at least one subsystem at any given time. In this paper, the switching regions are defined according to equation (5). At each instant, the active subsystem is chosen based on the minimum rule proposed by Kim et al. in [28].

The proposed switching law is further modified and as defined as :

$$\sigma(t) = i, \ if \ x(t) \in \tilde{\Lambda}_i, \ i \in N = \{1, 2, ..., l\}. \tag{9}$$

In this work, a stability analysis approach similar to that of [27] and [24] is used. Moreover, an additional hypothesis is introduced in order to facilitate the computation of the derivative of the Lyapunov–Krasovskii functional, extending the method beyond what was considered in [27] and [24].

2.3 Useful Lemmas

Before obtaining the main results, we first introduce some useful lemmas, which show a favorable scheme for the calculation and derivation of the main conclusions.

Lemma 1 *(Jensen's inequality) [27]. For any symmetric and positive definite matrix* $M \in \mathbb{R}^{n\times n}$*, positive scalar* λ*, and a vector function* $\omega(s) : [t-\lambda, t] \to \mathbb{R}^n$*, the following inequality holds :*

$$\lambda \int_{t-\lambda}^{t} \omega^T(s) M \omega(s)\, ds \geq \left[\int_{t-\lambda}^{t} \omega(s)\, ds\right]^T M \left[\int_{t-\lambda}^{t} \omega(s)\, ds\right] \tag{10}$$

Lemma 2 *(Schur complement) [29]. Let two symmetric matrices* $Q \in \mathbb{R}^{n\times n}$ *and* $R \in \mathbb{R}^{m\times m}$*, and any matrix* $S \in \mathbb{R}^{m\times n}$*, then the following conditions are equivalent :*

1. $\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} < 0.$
2. $R < 0, \quad Q - SR^{-1}S^T < 0.$
3. $Q < 0, \quad R - S^T Q^{-1} S < 0.$

Lemma 3 *[27]. For any positive definite matrix* $A \in \mathbb{R}^{n\times n}$ *(or negative definite matrix A) and symmetric matrix* $B \in \mathbb{R}^{n\times n}$*, there exists a small number* $\bar{\epsilon}$ *such that the following inequality holds for any* $\epsilon \in [0,\ \bar{\epsilon}]$ *:*

$$A - \epsilon B > 0 \ \ (< 0). \tag{11}$$

By applying the Leibniz-Newton formula, one gets the following form :

$$x(t - h(t)) = x(t) - \int_{t-h(t)}^{t} \dot{x}(s) ds. \tag{12}$$

3 Stability analysis and controller design

In order to establish a framework for stability analysis, the methods proposed by [28], [27] and [24] are essentially analytical techniques designed. Nevertheless, the systems analyzed in these references become unstable once the delay exceeds very small values. In contrast, the control strategy developed in this paper effectively addresses this limitation, ensuring stability even when the delay is way larger than the amounts have been calculated based on existing methods. The main aim of this work lies in reducing the entries of the A_{hi} matrices, thereby enabling a larger delay bound for ensuring system stability. To achieve this objective, the following controller is constructed :

$$u_i(t) = \beta_i A_{hi} \int_{t-h(t)}^{t} \dot{x}(s) ds \tag{13}$$

A_{hi} are defined in (1), β_i are some given positif scalars satisfying the following condition:

$$0 < \beta_i < 1$$

By taken into account the Leibniz-Newton formula (12) and considering control input as follows :

$$u_i(t) = \beta_i \Upsilon_i \int_{t-h(t)}^{t} \dot{x}(s) ds, \quad with \ \ \Upsilon_i \in \mathbb{R}^{n\times n}, \quad and \ \ 0 < \beta_i < 1, \quad i \in N. \tag{14}$$

Thus, the system (1) can be written in the following expression :

$$\dot{x}(t) = (A_i + A_{hi})x(t) - \int_{t-h(t)}^{t} (A_{hi} - \beta_i \Upsilon_i)\dot{x}(s) ds \tag{15}$$

Theorem 1 *By introducing a control strategy with the expression in equation (13), which can be written in the form* $u_i(t) = \beta_i \Upsilon_i[x(t) - x(t - h(t))]$, *the closed-loop system (15) is asymptotically stable if there exist positive definite and symmetric matrices P, Q, R and V* $\in \mathbb{R}^{n\times n}$, *matrices* X_i, Y_i, Z_i *and* T_i *with appropriate dimensions and scalar* $\bar{\tau}$ *that for any given* $\tau \in (0,\ \bar{\tau}]$ *and* $h < 1$ *such that the following inequalities hold,* $\forall i \in \mathrm{N} = \{1, 2, ..., l\}$ *:*

$$\Psi_i = \begin{bmatrix} \Psi_{11i} & \Psi_{12i} & \Psi_{13i} & \Psi_{14i} & \Psi_{15i} \\ * & \Psi_{22i} & \Psi_{23i} & \Psi_{24i} & 0 \\ * & * & \Psi_{33i} & \Psi_{34i} & 0 \\ * & * & * & \Psi_{44i} & \Psi_{45i} \\ * & * & * & * & \Psi_{55i} \end{bmatrix} < 0, \tag{16}$$

$$F^T P + PF < 0 \tag{17}$$

where $(*)$ denotes the symmetric part in a symmetric matrix and,

$$\begin{aligned}
&\Psi_{11i} = Sym\{F^T P - (1-\beta_i)PA_{hi} + X_i\} + Q + R - V, && \Psi_{12i} = V + Y_i^T, \\
&\Psi_{13i} = (1-\beta_i)PA_{hi} - X_i + Z_i^T, && \Psi_{14i} = -X_i + T_i^T, && \Psi_{15i} = \tau(A_i + A_{hi})^T V, \\
&\Psi_{22i} = -Q - V, && \Psi_{23i} = -Y_i, && \Psi_{24i} = -Y_i, && \Psi_{33i} = -(1-h)R - Z_i - Z_i^T, \\
&\Psi_{34i} = -Z_i - T_i^T, && \Psi_{44i} = -T_i - T_i^T, && \Psi_{45i} = -\tau(1-\beta_i)A_{hi}^T V, && \Psi_{55i} = -V.
\end{aligned}$$

Furthermore, $0 \le h(t) \le \tau$, $0 < \beta_i < 1$ and the matrix H in equations (5) and (6) can has the following expression:

$$H = -(F^T P + PF) = -\sum_{i=1}^{n} \alpha_i (A_i + A_{hi}), \quad \sum_{i=1}^{l} \alpha_i = 1.$$

The switching regions in (7) can putting into the following form :

$$\tilde{\Lambda}_i = \{x^T[(A_i + A_{hi})^T P + P(A_i + A_{hi}) + \tfrac{1}{\varsigma} H]x \le 0\}$$

Proof 1 *For systems (1) and (15), we construct an appropriate Lyapunov-Krasovsii functional as following :*

$$V(x(t)) = \sum_{i=1}^{4} V_i(x(t)), \tag{18}$$

where,

$$\begin{aligned}
&V_1(x(t)) = x^T(t)Px(t), && V_2(x(t)) = \textstyle\int_{t-h(t)}^{t} x^T(s)Rx(s)ds, \\
&V_3(x(t)) = \textstyle\int_{t-\tau}^{t} x^T(s)Qx(s)ds, && V_4(x(t)) = \tau \textstyle\int_{-\tau}^{0}\int_{t-\theta}^{t} \dot{x}^T(s)V\dot{x}(s)dsd\theta.
\end{aligned}$$

The derivative of the constructed Lyapunov-Krasovskii function can be calculated in the following form :

$$\dot{V}(x(t)) = \dot{V}_1(x(t)) + \dot{V}_2(x(t)) + \dot{V}_3(x(t)) + \dot{V}_4(x(t)) \tag{19}$$

Let calculate $\dot{V}_1(x(t))$

$$\begin{aligned}
\dot{V}_1(x(t)) &= \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t) \\
&= x^T(t)[(A_i + A_{hi})^T P + P(A_i + A_{hi})]x(t) - \textstyle\int_{t-h(t)}^{t} \dot{x}^T(s)(1-\beta_i)A_{hi}^T Px(t)ds \\
&\quad - \textstyle\int_{t-h(t)}^{t} x^T(t)(1-\beta_i)PA_{hi}\dot{x}(s)ds.
\end{aligned}$$

Assumption 1 *[24] It is assumed that* $\Upsilon_i = A_{hi}$

By using this assumption, the inequality (6) and the calculation of the second integral in the previous expression :

$$\int_{t-h(t)}^{t} x^T(t)(1-\beta_i)PA_{hi}\dot{x}(s)ds = x^T(t)(1-\beta_i)PA_{hi}\int_{t-h(t)}^{t}\dot{x}(s)ds$$
$$= (1-\beta_i)x^T(t)PA_{hi}[x(t)-x(t-h(t))]$$

The same method is used to the calculation of the first integral of $\dot{V}_1(x(t))$, we obtain :

$$\int_{t-h(t)}^{t}\dot{x}^T(s)(1-\beta_i)A_{hi}^TPx(t)ds = [\int_{t-h(t)}^{t}\dot{x}^T(s)ds](1-\beta_i)A_{hi}^TPx(t)$$
$$= (1-\beta_i)[x^T(t)-x^T(t-h(t))]A_{hi}^TPx(t)$$

Then, $\dot{V}_1(x(t))$ is summarized as follows :

$$\dot{V}_1(x(t)) = x^T(t)[Sym\{PF-(1-\beta_i)PA_{hi}\}]x(t) + (1-\beta_i)x^T(t)PA_{hi}x(t-h(t))$$
$$+ (1-\beta_i)x^T(t-h(t))A_{hi}^TPx(t)$$

Calculation of $\dot{V}_2(x(t))$

$$\dot{V}_2(x(t)) = x^T(t)Rx(t) - (1-\dot{h}(t))x^T(t-h(t))Rx(t-h(t))$$

By using the condition $\dot{h}(t) \le h < 1$, we have :

$$\dot{V}_2(x(t)) \le x^T(t)Rx(t) - (1-h)x^T(t-h(t))Rx(t-h(t))$$

Following the same methodology, the calculation of $\dot{V}_3(x(t))$ gives :

$$\dot{V}_3(x(t)) = x^T(t)Qx(t) - x^T(t-\tau)Qx(t-\tau)$$

Calculation of $\dot{V}_4(x(t))$

$$\dot{V}_4(x(t)) = \tau^2\dot{x}^T(t)V\dot{x}(t) - \tau\int_{t-\tau}^{t}\dot{x}^T(s)V\dot{x}(s)ds$$

By applying Jensen Lemma (1), one can gets :

$$-\tau\int_{t-\tau}^{t}\dot{x}^T(s)V\dot{x}(s)ds \le -(\int_{t-\tau}^{t}\dot{x}^T(s)ds)V(\int_{t-\tau}^{t}\dot{x}(s)ds) \le -[x^T(t)-x^T(t-\tau)]V[x(t)-x(t-\tau)].$$

The first term $\tau^2\dot{x}^T(t)V\dot{x}(t)$ of $\dot{V}_4(x(t))$ can be calculated easily.
Choosing also,

$$\xi^T(t) = \begin{bmatrix} x^T(t) & x^T(t-\tau) & x^T(t-h(t)) & \int_{t-h(t)}^{t}\dot{x}^T(s)ds \end{bmatrix}^T$$

From the above results, we obtain:

$$\dot{V}(x(t)) \le \xi^T(t)\Phi_i\xi(t) \tag{20}$$

where,

$$\Phi_i = \begin{bmatrix} \Phi_{11i} & \Phi_{12i} & \Phi_{13i} & \Phi_{14i} \\ * & \Phi_{22i} & \Phi_{23i} & \Phi_{24i} \\ * & * & \Phi_{33i} & \Phi_{34i} \\ * & * & * & \Phi_{44i} \end{bmatrix}, \tag{21}$$

and,

$$\Phi_{11i} = Sym\{PF-(1-\beta_i)PA_{hi}\} + Q + R - V + \tau^2(A_i+A_{hi})^TV(A_i+A_{hi}),$$
$$\Phi_{12i} = V, \quad \Phi_{13i} = (1-\beta_i)PA_{hi}, \quad \Phi_{14i} = -\tau^2(1-\beta_i)(A_i+A_{hi})^TVA_{hi}, \quad \Phi_{22i} = -Q-V,$$
$$\Phi_{23i} = 0, \quad \Phi_{24i} = 0, \quad \Phi_{33i} = -(1-h)R, \quad \Phi_{34i} = 0, \quad \Phi_{44i} = \tau^2(1-\beta_i)^2A_{hi}^TVA_{hi}.$$

Introducing now some Free-Weighting matrices, such technique has been used in a significant number of recent works in the literature like [7], [30] and others. then, the following

equations and inequalities can be utilized to further reduce the conservativeness of the stability criterion. Notably, the equation derived from the Leibniz-Newton formula produces, in practice, a result that equal to zero.

$$2 \times [x^T(t)X_i + x^T(t-\tau)Y_i + x^T(t-h(t))Z_i + \int_{t-h(t)}^{t} \dot{x}^T(s)ds T_i] \times [x(t) - x(t-h(t)) - \int_{t-h(t)}^{t} \dot{x}(s)ds] = 0 \tag{22}$$

where, X_i, Y_i, Z_i and T_i are unknown matrices with appropriate dimensions.

Assuming (21) and (22), one can gets :

$$\dot{V}(x(t)) \leq \xi^T(t)\bar{\Phi}_i\xi(t) \tag{23}$$

Where,

$$\bar{\Phi}_i = \begin{bmatrix} \bar{\Phi}_{11i} & \bar{\Phi}_{12i} & \bar{\Phi}_{13i} & \bar{\Phi}_{14i} \\ * & \bar{\Phi}_{22i} & \bar{\Phi}_{23i} & \bar{\Phi}_{24i} \\ * & * & \bar{\Phi}_{33i} & \bar{\Phi}_{34i} \\ * & * & * & \bar{\Phi}_{44i} \end{bmatrix}, \tag{24}$$

$\bar{\Phi}_{11i} = Sym\{PF - (1-\beta_i)PA_{hi} + X_i\} + Q + R - V + \tau^2(A_i + A_{hi})^T V(A_i + A_{hi})$,
$\bar{\Phi}_{12i} = V + Y_i^T, \bar{\Psi}_{13i} = (1-\beta_i)PA_{hi} - X_i + Z_i^T, \bar{\Phi}_{14i} = -\tau^2(1-\beta_i)(A_i + A_{hi})^T VA_{hi} - X_i + T_i^T$,
$\bar{\Phi}_{22i} = -Q - V, \quad \Psi_{23i} = -Y_i, \quad \Psi_{24i} = -Y_i, \quad \bar{\Phi}_{33i} = -(1-h)R - Z_i - Z_i^T$,
$\Psi_{34i} = -Z_i - T_i^T, \quad \bar{\Phi}_{44i} = \tau^2(1-\beta_i)^2 A_{hi}^T VA_{hi} - T_i - T_i^T$.

To guarantee the asymptotically stability of system (1) and (15), and by using the suggested controller in (13), $\bar{\Phi}_i$ should be negative definite, that is :

$$\bar{\Phi}_i < 0 \tag{25}$$

Applying Schur complement in Lemma (2) to the matrices $\bar{\Phi}_i$, we found exactly the results of theorem 1, namely the LMIs (16) with the condition (16) must be verified.

Now, if Ψ_i be negative definite $\Psi_i < 0$, $\bar{\Phi}_i$ will be negative definite too ($\bar{\Phi}_i < 0$). But it was proved in Theorem 1, there exist $\bar{\tau}$ that for any given $\tau \in (0, \bar{\tau}]$, $\Psi_i < 0$. This result shows that system (1) or (15) is asymptotically stable. The proof is completed.

The architecture of the proposed controller is depicted in the block diagram shown in Figure 1.

Remark 1 *The existence of overlapping zones among the switching regions of the subsystems leads to a deceleration of the switching dynamics at the boundaries of the switching regions. As a consequence, the transition from one controller to another becomes slower. This phenomenon will be shown in Example 1.*

Remark 2 *State-dependent switching rules considered in this paper, can provide better performance by aligning the switching logic with the system's current behavior, but they may increase design complexity and implementation cost. Moreover, in the proof of theorem 1, we did not use any model transformation of system, which often leads to a large computational load and increases the complexity. We also used an appropriate multiple L-K fonctional slightly different to that used several works in the literature such as [24], [28], [27] and others. We have introduced some slack variables by using Free-weighting-Matrix to give more relaxation for the studied system.*

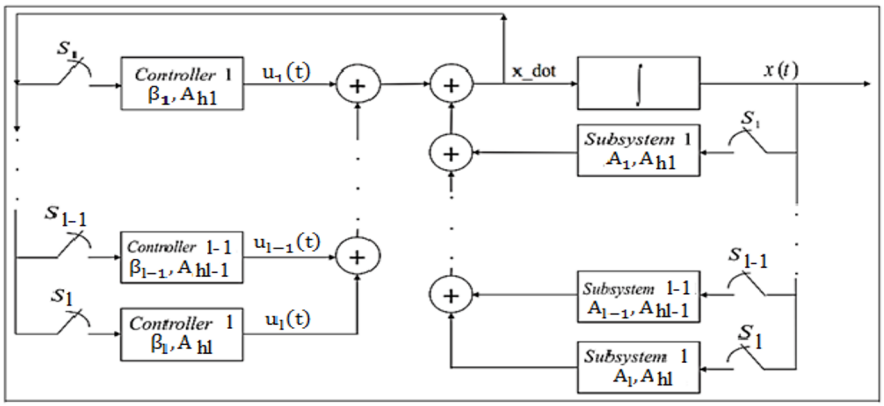

Figure 1. Block diagram of proposed controller for considered system in (1).

4 Numerical examples and simulation results

In this section, some numerical examples are given from literature to verify the effectiveness and the validity of the suggested method.

Example 1 *(Switching between unstable subsystems) example 1 of [24]. Consider the following linear continuous-time switched system with time-varying delay as :*

$$\dot{x}(t) = A_i x(t) + A_{hi} x(t - h(t)) + u_i(t), \sigma = i \in \mathrm{N} = \{1, 2\} \tag{26}$$

Where,

$$(A_1,\ A_{h1}) = \left(\begin{bmatrix} -2 & -2 \\ -20 & -2 \end{bmatrix}, \begin{bmatrix} -1 & -7 \\ 23 & 6 \end{bmatrix}\right), \qquad (A_2,\ A_{h2}) = \left(\begin{bmatrix} -2 & 10 \\ -4 & -2 \end{bmatrix}, \begin{bmatrix} 4 & -5 \\ 1 & -8 \end{bmatrix}\right).$$

We can verify easily that both (A_1, A_{h1}) and (A_2, A_{h2}) are unstable. Taken further $\alpha_1 = 0.6$ and $\alpha_2 = 0.4$. Then, the Hurwitz convex combination F, will be equal to:

$$F = 0.6(A_1 + A_{h1}) + 0.4(A_2 + A_{h2}) = \begin{bmatrix} -1 & -1 \\ 0.6 & -1.6 \end{bmatrix}$$

Let consider for constant delay, the following parameters $\beta_1 = \beta_2 = 0.98$, by solving LMIs (16)-(17) in theorem 1, provides the follow upper limit of the allowable delay : $\bar{\tau} \geq 3.5650$ and :

$$P = \begin{bmatrix} 10.8747 & -0.8555 \\ -0.8555 & 9.5266 \end{bmatrix}, \qquad F^T P + PF = -H = \begin{bmatrix} -22.7759 & -2.9344 \\ -2.9344 & -28.7742 \end{bmatrix}$$

Also, the suggested controllers $u_1(t)$ and $u_2(t)$ are equal to :

$$u_1(t) = \begin{bmatrix} -0.9800 & -6.8600 \\ 22.5400 & 5.8800 \end{bmatrix} \int_{t-h(t)}^{t} \dot{x}(s)ds,\ u_2(t) = \begin{bmatrix} 3.9200 & -4.9000 \\ 0.9800 & -7.8400 \end{bmatrix} \int_{t-h(t)}^{t} \dot{x}(s)ds \tag{27}$$

By employing the matrices given A_1, A_{h1}, A_2 and A_{h2}, together with the computed matrices P and H in Example 1, and substituting them into equation (7) with $\varsigma = 2$, the switching regions are otained. The corresponding switching law is established according to equation (9). Therefore, based on Theorem 1, system (26) is asymptotically stable for a delay bound

$\bar{\tau} \geq 3.5650$, under switching law in (9). However, by employing the approach in [24], the obtained delay bound is $\bar{\tau} \leq 0.849$, and in [27], when delay is constant, the value of this delay bound is $\bar{\tau} = 0.0202$, for the same example, when $u_i(t) = 0$. In the this note, the application of the proposed controller resulted in a substantially larger admissible delay bound $\bar{\tau} \geq 3.5650$, which significantly improves bounds than the selected results using other various approaches and demonstrates that the developed method in this paper provides less conservative results than that some recent ones in the literature.

With the initial condition $\phi(\theta) = \begin{bmatrix} -3 & 2 \end{bmatrix}^T$. By applying the control inputs $u_1(t)$ and $u_2(t)$ in equation (27) to the studied system, the state trajectories of system (26) with constant delay $h(t) = 3.5650$ are illustrated in Figure 2, it is observed that both of the trajectories converge quickly to the origin. The corresponding control signals $u_1(t)$ and $u_2(t)$ for each subsystem are presented in Figure 3, while the switching signal is depicted in Figure 4. This results illustrates the importance of the designed control in improving the performance of dynamic switched system.

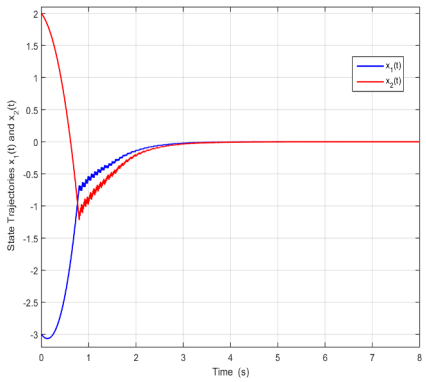

Figure 2. State Trajectories of system (26) with constant delay $h(t) = 3.5650$ and selected switching law.

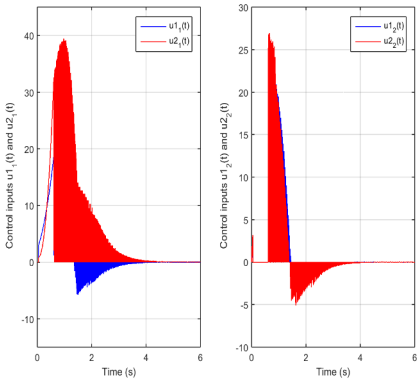

Figure 3. Control inputs of subsystem 1 and subsystem 2, with constant delay $h(t) = 3.5650$ and selected switching law.

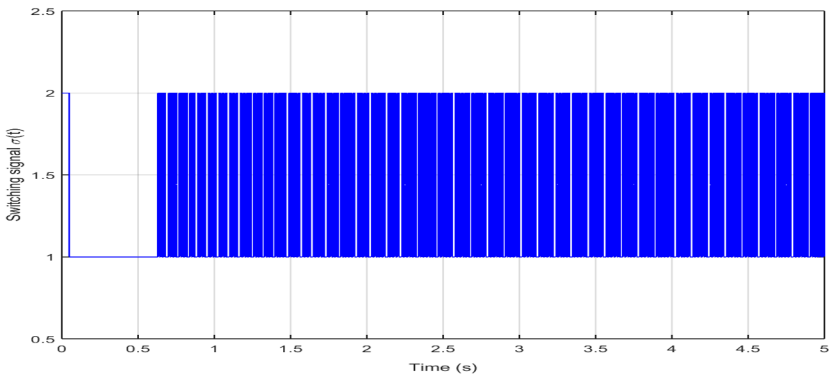

Figure 4. Switching signal of system (26) with constant delay $h(t) = 3.5650$.

To illustrate the impact of overlapping regions generated by (7) with $\varsigma = 2$, the state trajectories of system (26) over the time interval [1.5, 1.8] are displayed in Figure 5. For comparison, when $\varsigma = 1$, corresponding to the case without overlapping regions and switching areas defined by (6), the state trajectories in the same interval are shown in Figure 6. A comparison of the two figures clearly indicates that fast switching occurs in the absence of overlapping areas.

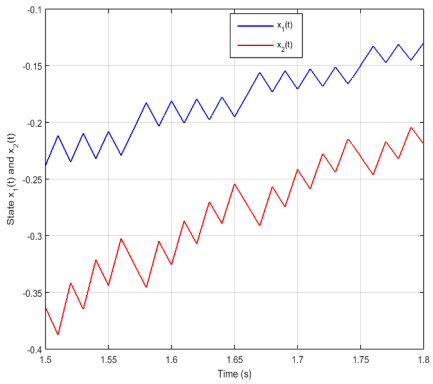

Figure 5. State Trajectories of system (26), with $\varsigma = 2$.

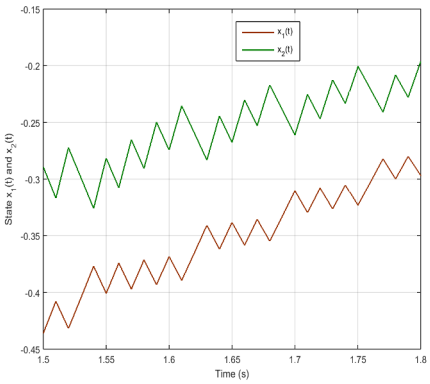

Figure 6. State Trajectories of system (26), with $\varsigma = 1$.

On the other hand, by using the same parameters mentioned in [24], especially the constant delay as $h(t) = 0.849$, $\beta_1 = \beta_2 = 0.98$, and the n solving LMIs (16) and (17), on can gets:

$$P = \begin{bmatrix} 10.3469 & -0.8495 \\ -0.8495 & 9.5950 \end{bmatrix}, \qquad F^T P + PF = -H = \begin{bmatrix} -21.7132 & -2.3811 \\ -2.3811 & -29.0051 \end{bmatrix}.$$

The simulation results with the initial condition $\phi(\theta) = \begin{bmatrix} -3 & 2 \end{bmatrix}^T$ are given as follows: Figure 7 depicts the state trajectories of system (26) under a constant delay $h(t) = 0.849$ showing convergence to the origin. The associated control signal $u_1(t)$ and $u_2(t)$ are displayed in Figure 8, and the corresponding switching signal is illustrated in Figure 9.

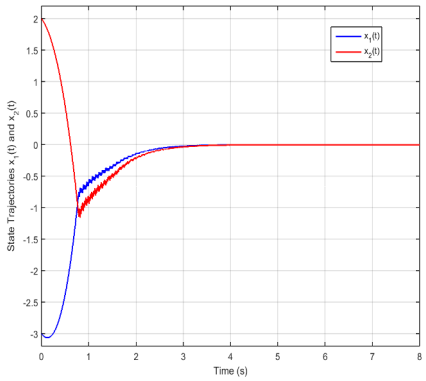

Figure 7. State Trajectories of system (26) with constant delay $h(t) = 0.849$ and selected switching law.

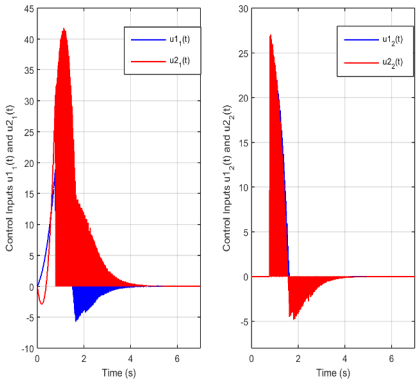

Figure 8. Control inputs of subsystem 1 and subsystem 2, with constant delay $h(t) = 0.849$ and selected switching law.

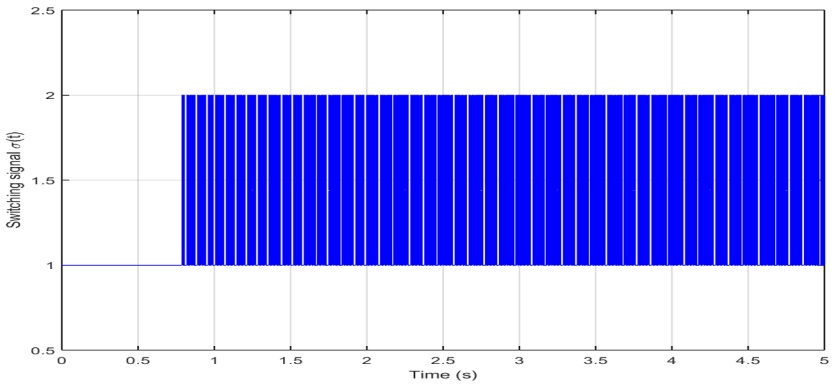

Figure 9. Switching signal of system (26) with constant delay $h(t) = 0.849$.

To evaluate the effectiveness of the proposed controller, the time-varying delay $h(t) = 0.429 - 0.42\sin(t)$ is adopted, as in [24]. The maximum delay bound is clearly 0.849, with $h = 0.42$. By setting $\beta_1 = \beta_2 = 0.98$ and solving the LMIs in (16) and (17), we obtain :

$$P = \begin{bmatrix} 14.1096 & -0.8912 \\ -0.8912 & 12.9176 \end{bmatrix}, \qquad F^T P + PF = -H = \begin{bmatrix} -29.2887 & -4.0418 \\ -4.0418 & -39.5540 \end{bmatrix}$$

Consider the initial condition as $\phi(\theta) = \begin{bmatrix} -3 & 2 \end{bmatrix}^T$, $\varsigma = 2$ and $u_1(t)$, $u_2(t)$ are presented in (27) and the switching signal is introduced in (9). So, by using time-varying delay $h(t) = 0.429 - 0.42\sin(t)$, the State trajectories of system (26) are depicted in Figure 10. Controllers $u_1(t)$ and $u_2(t)$ are shown in Figure 11 and the switching law is shown in Figure 12.

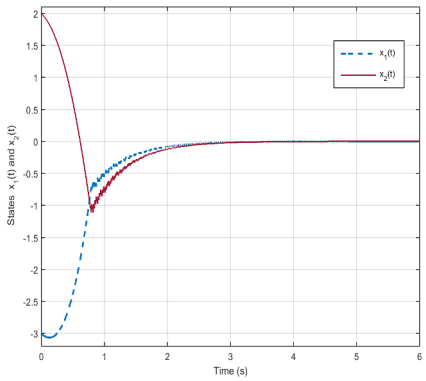

Figure 10. State Trajectories of system (26) with time-varying delay $h(t) = 0.429 - 0.42\sin(t)$ and selected switching law.

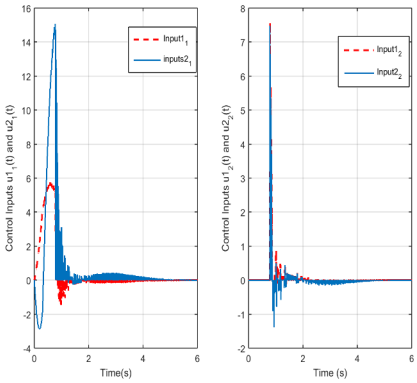

Figure 11. Control inputs of subsystem 1 and subsystem 2, with time-varying delay $h(t) = 0.429 - 0.42\sin(t)$ and considered switching law.

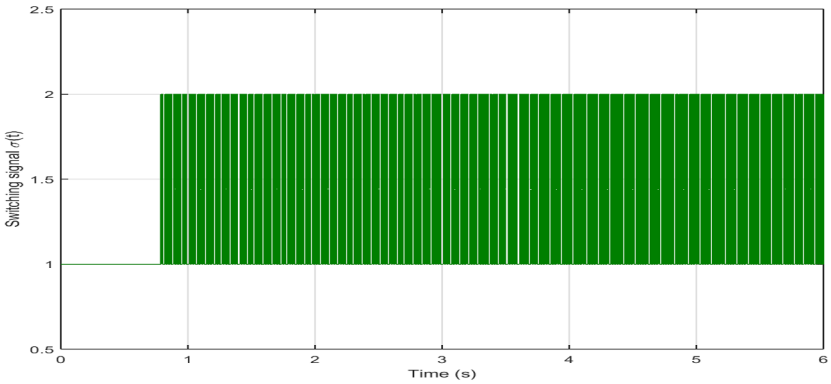

Figure 12. Switching signal of system (26) with time-varying delay $h(t) = 0.429 - 0.42\sin(t)$.

By comparing the results illustrated in Figures 8 and 9 for constant delay case, and Figures 10 and 11 for time-varying delay case, with those reported in [24] under the same parameter settings, it can be observed that the state trajectories obtained with the proposed method converge more rapidly to zero and exhibit reduced oscillations compared to those of the reference approach. Moreover, the controllers demonstrate superior performance.

5 Conclusion

This paper has presented a controller design for a class of linear switched systems with time-varying delays. The stability analysis was carried out using the proposed Lyapunov–Krasovskii functional, with the results formulated in terms of linear matrix inequalities. The existence of the proposed controller relies on the presence of a Hurwitz convex combination. Compared to existing approaches, the controller achieves stability over a larger admissible delay bound. Its main advantage lies in overcoming the limitations associated

with small delay bounds, thereby ensuring stability even under significantly larger delays. Furthermore, the switching rule adopted in this study mitigates rapid subsystem switching, which in turn reduces sliding motions in the state trajectories. While the creation of overlapping regions at the boundaries of the switching domains helps to attenuate chattering, future investigations could explore the applicability of similar approaches for switched systems with large delays under alternative switching strategies. It is also important to note that, in this work, all state variables were assumed to be available. Since this assumption may not hold in practice due to measurement constraints or the high cost of sensing devices, future research should consider the development of observer-based methods to estimate the state variables.

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